Probability and Random Variables (ECE313/ECE317)

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Random variables
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Probability: Random variables

- Random variable: Discrete / Continuous.
- Expectation (mean), Variance, standard deviation.
- Mathematical background
- Discrete Random variables - Probability mass function (PMF).
  - Uniform
  - Bernoulli
  - Binomial
  - Geometric
  - Poisson
- Continuous Random variables - Probability density function (PDF).
  - Uniform
  - Exponential
  - Normal
- Multi-Random variable: Covariance, Correlation.

Probability: Random variables

Random variable: Usually written X;
Is a variable (or a function) whose possible values are numerical quantities that take random values (from a sample set) corresponding to outcomes of a random phenomena.

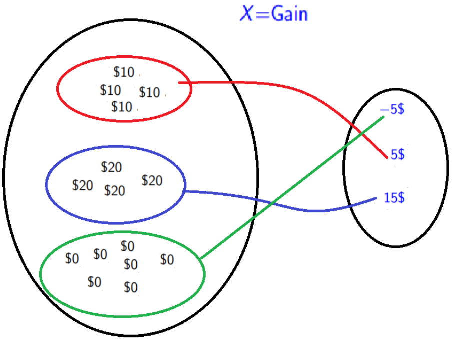
$$\begin{array}{rccc} X: & \Omega & \to & \mathbb{R} \\ & \omega & \to & X(\omega) = x \end{array}$$

Example1:
A raffle consists of 15 tickets. There are 4 tickets of \$10, 4 tickets of \$20 and 7 losing tickets (\$0). You have to buy a ticket that costs \$5.

X: The random variable associating to the **player's gain**

- Possible cases=$\{\$0, \$10, \$20\}$
- X=Gain=$\{-5\$, 5\$, 15\$\}$

Probability: Random variables



$$\Omega = \text{Universe} = 15 \text{ tickets} \rightarrow \quad X: \quad \Omega \rightarrow \mathbb{R}$$
$$\omega \rightarrow X(\omega) = \{-5, 10, 15\}$$

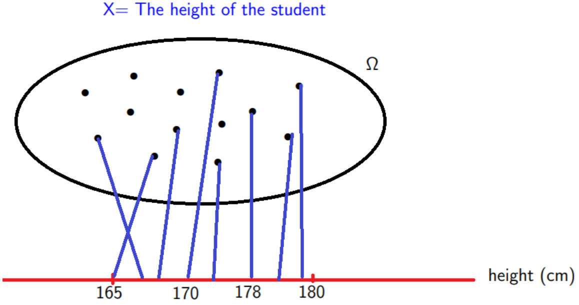
$$\mathbb{P}(X = -5) = \frac{7}{15} = \mathbb{P}(\omega = 0), \; \mathbb{P}(X = 5) = \frac{4}{15} = \mathbb{P}(\omega = 10), \; \mathbb{P}(X = 15) = \frac{4}{15} = \mathbb{P}(\omega = 20).$$

- A random variable if a function of the outcomes of some experiment

Probability: Random variables

Example2:

- $\Omega = \{$Students of the ECE-313 class$\}$
- Experiment:"Choose randomly a student and measure the height of that student "
- X= The height of the student



$$\mathbb{P}(160 \leq X \leq 180) = ?$$

Probability: Random variables

▷ A random variable can be **discrete** or **continuous**.

- Discrete random variable: Takes only countable numbers.
- Number of children in a family.
- Number of patients in a doctor surgery.
- Number of defective light bulbs in a box.
- The birthday of a student chosen randomly.

- Continuous random variable: Takes its values in an interval.
- The weight of an animal taken at random in a park.
- The temperature.
- The distance.

Probability: Random variables

- Notation:

$$\begin{aligned} X: \quad \Omega &\rightarrow \mathbb{R} \\ \omega &\rightarrow X(\omega) = x \end{aligned}$$

$$\mathbb{P}_X(x) = \mathbb{P}(X = x) = \mathbb{P}(\omega \in \Omega : \quad X(\omega) = x)$$

$$0 \leq \mathbb{P}_X(x) \leq 1, \qquad \sum_x \mathbb{P}_X(x) = 1$$

- Probability mass function (PMF):

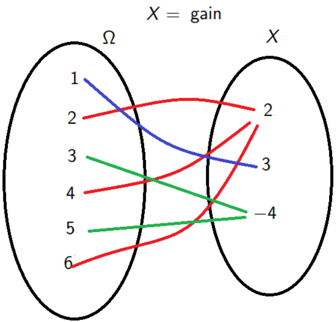
The probability mass function is $\mathbb{P}_X(.)$ and called the **probability distribution** or the **probability law**

Probability: Random variables

Example:

Flipping a die of 6 faces. If the result is even, we earn \$2, if the result is 1, we earn \$3, and if the result is 3 or 5, we lose \$4. What is the probability distribution of the random variable X which gives the gain of this game.

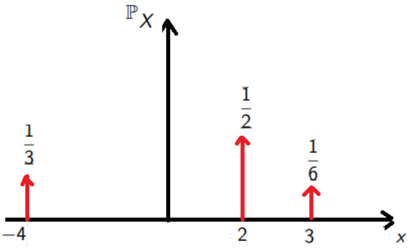
- $\Omega = \{1, 2, 3, 4, 5, 6\}$
- $X = \{\$2, \$3, \$-4\}$



Probability: Random variables

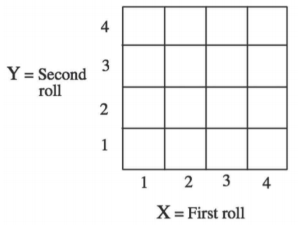
x_i	2	3	-4
$\mathbb{P}(X = x_i)$	$\mathbb{P}(X = 2) = \frac{1}{2}$	$\mathbb{P}(X = 3) = \frac{1}{6}$	$\mathbb{P}(X = -4) = \frac{1}{3}$

$$\mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \mathbb{P}(X = -4) = \frac{1}{2} + \frac{1}{6} + \frac{1}{3} = 1$$



Probability: Examples

Example: Two rolls of a tetrahedral die



$$\Omega = \{\text{All } (X, Y)\} = \{(1,1), (1,2), \dots (4,4)\} \Rightarrow 16 \text{ elements}$$

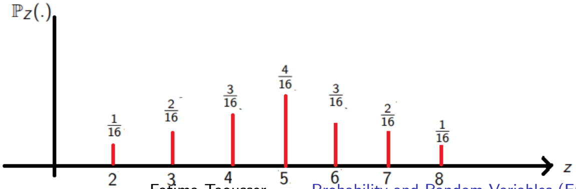
$$\mathbb{P}(\text{each } (X, Y)) = \frac{1}{16}$$

Random variables: Examples

Let the random variable $Z = X + Y$. Find the probability distribution $\mathbb{P}_Z(.)$

$$Z = \{2, 3, 4, 5, 6, 7, 8\}$$

$$\{Z = 2\} = \{(1,1)\} \rightarrow \mathbb{P}(Z = 2) = \frac{1}{16}$$
$$\{Z = 3\} = \{(1,2),(2,1)\} \rightarrow \mathbb{P}(Z = 3) = \frac{2}{16}$$
$$\{Z = 4\} = \{(1,3),(3,1),(2,2)\} \rightarrow \mathbb{P}(Z = 4) = \frac{3}{16}$$
$$\{Z = 5\} = \{(1,4),(4,1),(2,3),(3,2)\} \rightarrow \mathbb{P}(Z = 5) = \frac{4}{16}$$
$$\{Z = 6\} = \{(2,4),(4,2),(3,3)\} \rightarrow \mathbb{P}(Z = 6) = \frac{3}{16}$$
$$\{Z = 7\} = \{(3,4),(4,3)\} \rightarrow \mathbb{P}(Z = 7) = \frac{2}{16}$$
$$\{Z = 8\} = \{(4,4)\} \rightarrow \mathbb{P}(Z = 8) = \frac{1}{16}$$



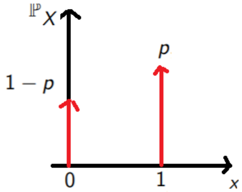
Random variables

Example: Tossing an unfair coin, such that $\mathbb{P}(H) = p$ and $\mathbb{P}(T) = 1 - p$

- $\Omega = \{H, T\}$

- Let $X = \begin{cases} 1; & \text{if we will get H} \\ 0; & \text{if we will get T} \end{cases}$

$$\mathbb{P}(X = 1) = p, \qquad \mathbb{P}(X = 0) = 1 - p$$



Random variables: Expectation

▷ Expectation: Is the weighted average of all possible values of the random variable X and it is denoted by $\mathbb{E}[X]$ → The weight here means the probability of each specific value of the random variable.

$$\begin{aligned} X : \quad \Omega &\rightarrow \mathbb{R} \\ \omega &\rightarrow X(\omega) = x \end{aligned} \quad \rightarrow \quad \mathbb{E}[X] = \sum_i x_i \mathbb{P}(X = x_i)$$

Example: The following probability distribution tells us the probability that a certain soccer team scores a certain number of goals in a given game

Goals (X)	Probability P(X)
0	0.18
1	0.34
2	0.35
3	0.11
4	0.02

$$\mathbb{E}[X] = \sum_i x_i \mathbb{P}(X = x_i) = (0 \times 0.18) + (1 \times 0.34) + (2 \times 0.35) + (3 \times 0.11) + (4 \times 0.02) = 1.45$$

This represents the expected number of goals (1.45 goals) that the team will score in any given game.

Random variables: Expectation

Expectation vs the mean (or average):

We typically calculate the mean after we have actually collected raw data. Let x_i: Raw data values, and n: Sample size $\rightarrow$ Mean $= \sum_{i=1}^{n} \frac{x_i}{n}$

Example:

Suppose we record the number of goals that a soccer team scores in 15 different games:
Goals Scored: $\{1, 1, 0, 2, 2, 1, 0, 3, 1, 1, 1, 2, 4, 3, 1\}$

$$\text{Mean} = \frac{1+1+0+2+2+1+0+3+1+1+1+2+4+3+1}{15} = 1.533 \text{ goals.}$$

This represents the mean number of goals scored per game by the team.

Remark:

• The expectation value of a random variable is the average that you expect to see in a large number of independent repetitions of the experiment.

• The expected value is not one of the possible values (or outcomes). However, if you average the values of a large number of experiments, the result approaches the expected value.

Random variables: Expectation

Example1: You are plying a game where you can earn \$4 with probability $\frac{1}{4}$,\$1 with probability $\frac{1}{2}$ and you can loss \$2 with probability $\frac{1}{4}$ $\rightarrow X = \{\$4, \$1, -\$2\}$

x_i	4	1	-2
$\mathbb{P}(X = x_i)$	$\mathbb{P}(X = 4) = \frac{1}{4}$	$\mathbb{P}(X = 1) = \frac{1}{2}$	$\mathbb{P}(X = -2) = \frac{1}{4}$

How much do you expect to win by plying this game many times?

$$\mathbb{E}[X] = (\frac{1}{4} \times \$4) + (\frac{1}{2} \times \$1) + (\frac{1}{4} \times -\$2) = \$1$$

- If I will play this game many times, I'm expecting to win \$1

Random variables-Expectation

Example2:

Let an experiment consist of tossing a fair coin three times. Let X **denote the number of Heads**. Then the possible values of X are $\{0, 1, 2, 3\}$.

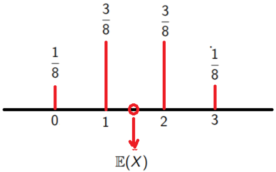
$$\Omega = \{\underbrace{HHH}_{X=3}, \underbrace{HHT}_{X=2}, \underbrace{HTH}_{X=2}, \underbrace{HTT}_{X=1}, \underbrace{THH}_{X=2}, \underbrace{THT}_{X=1}, \underbrace{TTH}_{X=1}, \underbrace{TTT}_{X=0}\}$$

The corresponding probabilities are:

$$\mathbb{P}(X=0) = \frac{1}{8}, \quad \mathbb{P}(X=1) = \frac{3}{8}, \quad \mathbb{P}(X=2) = \frac{3}{8}, \quad \mathbb{P}(X=3) = \frac{1}{8}$$

Thus, the expected value of X is

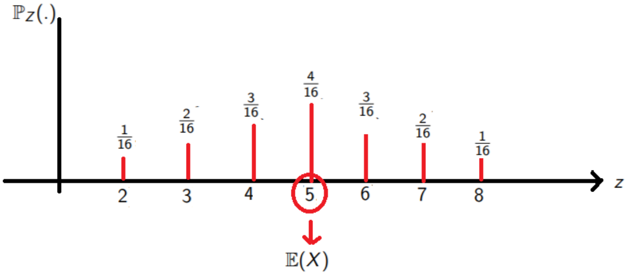
$$\mathbb{E}[X] = (0 \times \frac{1}{8}) + (1 \times \frac{3}{8}) + (2 \times \frac{3}{8}) + (3 \times \frac{1}{8}) = \frac{3}{2} = 1.5.$$



Random variables-Expectation

Example3: Two rolls of a tetrahedral die: $X = \{1, 2, 3, 4\}$ and $Y = \{1, 2, 3, 4\}$
Let the random variable $Z = X + Y$.

z_i	2	3	4	5	6	7	8
$\mathbb{P}(Z = z_i)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$



$$\mathbb{E}[Z] = (2 \times \frac{1}{16}) + (3 \times \frac{2}{16}) + (4 \times \frac{3}{16}) + (5 \times \frac{4}{16}) + (6 \times \frac{3}{16}) + (7 \times \frac{2}{16}) + (8 \times \frac{1}{16}) = 5.$$

Random variables

- Properties of the expectation:

- If $X \geq 0$, then $\mathbb{E}[X] \geq 0$

$\mathbb{E}[X] = \sum_i x_i \; \mathbb{P}(X = x_i)$. Since all $x_i \geq 0$ and $\mathbb{P}(.) \geq 0 \;\Rightarrow\; \mathbb{E}(X) \geq 0$

- If $a \leq X \leq b$, then $a \leq \mathbb{E}[X] \leq b$

$$a \leq x_i \leq b \Rightarrow a \; \mathbb{P}(X = x_i) \leq x_i \; \mathbb{P}(X = x_i) \leq b \; \mathbb{P}(X = x_i)$$
$$\Rightarrow \sum_i a \; \mathbb{P}(X = x_i) \leq \sum_i x_i \; \mathbb{P}(X = x_i) \leq \sum_i b \; \mathbb{P}(X = x_i)$$
$$\Rightarrow \; a \underbrace{\sum_i \mathbb{P}(X = x_i)}_{=1} \leq \underbrace{\sum_i x_i \; \mathbb{P}(X = x_i)}_{\mathbb{E}[X]} \leq b \underbrace{\sum_i \mathbb{P}(X = x_i)}_{=1} \Rightarrow a \leq \mathbb{E}[X] \leq b$$

- If $X = c$ is a constant value, then $\mathbb{E}[c] = c$

$$\mathbb{E}[X] = \sum_i x_i \mathbb{P}(X = x_i) = \sum_i c \; \mathbb{P}(X = c) = c \underbrace{\sum_i \mathbb{P}(X = c)}_{=1} = c$$

Random variables

- $\mathbb{E}[aX + b] = a\mathbb{E}[x] + b$ (linearity property)

$$\mathbb{E}[aX + b] = \sum_i (ax_i + b)\ \mathbb{P}(X = x_i) = \sum_i [ax_i\ \mathbb{P}(X = x_i) + b\ \mathbb{P}(X = x_i)]$$

$$= a\underbrace{\sum_i x_i\ \mathbb{P}(X = x_i)}_{=\mathbb{E}[X]} + b\underbrace{\sum_i\ \mathbb{P}(X = x_i)}_{=1} = a\mathbb{E}[X] + b$$

- Example: Let X be the salary of some employers, and $\mathbb{E}[X] = 1500$ be the

expected salary. Let Y be a new salary, s.t $Y = 2X + 100$. what is the expectation of this new salary?

$$\mathbb{E}[Y] = \mathbb{E}[2X + 100] = 2\ \mathbb{E}(X) + 100 = 3100$$

Random variables

$$\mathbb{E}[g(X)] = \sum_i g(x_i)\, \mathbb{P}(X = x_i)$$

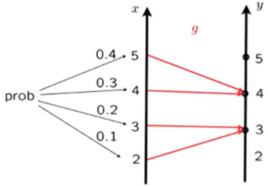
Example 1: Let the random variable $X = \{2, 3, 4, 5\}$, such that

$$\mathbb{P}(X = 2) = 0.1, \quad \mathbb{P}(X = 3) = 0.2, \quad \mathbb{P}(X = 4) = 0.3, \quad \mathbb{P}(X = 5) = 0.4$$

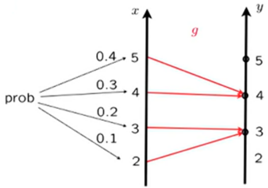
Let $Y = g(X)$ be another random variable determined by the original random variable X such that $Y = g(X) = \{4, 3\}$, as it is shown in the figure below, such that

- $g(5) = g(4) = 4 \to \mathbb{P}(Y = 4) = \mathbb{P}(X = 5) + \mathbb{P}(X = 4) = 0.4 + 0.3 = 0.7$
- $g(3) = g(2) = 3 \to \mathbb{P}(Y = 3) = \mathbb{P}(X = 2) + \mathbb{P}(X = 3) = 0.1 + 0.2 = 0.3$

- What is the expectation of Y?



Random variables



Method 1:

$$\mathbb{E}[Y] = \sum_i y_i \; \mathbb{P}(Y = y_i) = 3\ (0.3) + 4\ (0.7) = 3.7$$

Method 2:

$$\mathbb{E}[g(X)] = \sum_i g(x_i)\ \mathbb{P}(X = x_i) = g(2)\mathbb{P}(X = 2) + g(3)\mathbb{P}(X = 3) + g(4)\mathbb{P}(X = 4) + g(5)\mathbb{P}(X = 5)$$

$$= 3\ (0.1) + 3\ (0.2) + 4\ (0.3) + 4\ (0.4) = 3.7$$

Random variables

Example 2: Let the random variable $X = \{2, 3, 4, 5\}$, such that

$$\mathbb{P}(X = 2) = 0.1, \quad \mathbb{P}(X = 3) = 0.2, \quad \mathbb{P}(X = 4) = 0.3, \quad \mathbb{P}(X = 5) = 0.4$$

1) What is the expectation of $Y = g(X) = X^2$?
2) Compute in two different ways the expectation of $Y = g(X) = 3X + 1$.

Solution:

$$1)\mathbb{E}[Y] = \mathbb{E}[X^2] = \sum_i x_i^2 \mathbb{P}(X = x_i) = (2^2 \times 0.1) + (3^2 \times 0.2) + (4^2 \times 0.3) + (5^2 \times 0.4) = 17$$

2) **Method 1:**

$$\mathbb{E}[Y] = \mathbb{E}[3X+1] = \sum_i (3x_i+1)\mathbb{P}(X = x_i) = [(3(2)+1)\times 0.1]+[(3(3)+1)\times 0.2]+[(3(4)+1)\times 0.3]$$

$$+[(3(5) + 1) \times 0.4] = 13$$

Method 2: We can use the linearity property

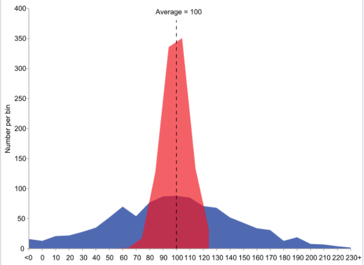
$$\mathbb{E}[Y] = \mathbb{E}[3X + 1] = 3\mathbb{E}[X] + 1 = 3\underbrace{[(2 \times 0.1) + (3 \times 0.2) + (4 \times 0.3) + (5 \times 0.4)]}_{=4} + 1 = 13$$

Random variables: Variance

▷ The variance of a random variable X is often written as $V(X)$, and it is defined by:

$$\boxed{V(X) = \sum_i [x_i - \mathbb{E}[X]]^2 \; \mathbb{P}(X = x_i)}$$ → The mean of the square of distances from the expectation

- The **variance** measures the spread of numerical variables and determines the degree to which the values differ from the mean. .
- If many values are close to the mean then the variance is small and if many values are far from the mean then the variance is large.
- In the following example: The red population has $\mathbb{E} = 100, \quad V = 100$, and the blue population has $\mathbb{E} = 100, \quad V = 2500$



Example of samples from two populations with the same mean but

Random variables

- Variance: The variance can be formulated as

$$\begin{aligned} V(X) &= \sum_i [(x_i - \mathbb{E}[X])^2 \; \mathbb{P}(X = x_i)] \\ &= \sum_i [(x_i^2 - 2x_i \; \mathbb{E}(X) + \mathbb{E}^2[X]) \; \mathbb{P}(X = x_i)] \\ &= \sum_i x_i^2 \; \mathbb{P}(X = x_i) - 2 \sum_i (x_i \; \mathbb{E}[X]) \; \mathbb{P}(X = x_i) + \sum_i \mathbb{E}^2[X] \; \mathbb{P}(X = x_i) \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X] \underbrace{\sum_i x_i \; \mathbb{P}(X = x_i)}_{\mathbb{E}[X]} + \mathbb{E}^2[X] \underbrace{\sum_i \mathbb{P}(X = x_i)}_{=1} \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X] \; \mathbb{E}[X] + \mathbb{E}^2[X] = \mathbb{E}[X^2] - \mathbb{E}^2[X] \end{aligned}$$

- The variance is the expectation of the squared deviation of a random variable from its mean.

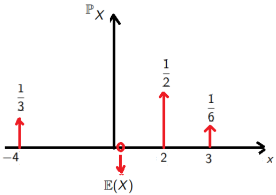
$$\boxed{\mathrm{V(X)} = \sum_i [x_i - \mathbb{E}[X]]^2 \; \mathbb{P}(X = x_i) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}^2[X]}$$

Random variables: Variance

- Example: Random variable X has the following PMF (or probability distribution)

x_i	2	3	-4
$\mathbb{P}(X = x_i)$	$\mathbb{P}(X=2) = \frac{1}{2}$	$\mathbb{P}(X=3) = \frac{1}{6}$	$\mathbb{P}(X=-4) = \frac{1}{3}$

$$\mathbb{E}[X] = \sum_x x_i\,\mathbb{P}(X = x_i) = 2\,\mathbb{P}(X=2) + 3\,\mathbb{P}(X=3) - 4\,\mathbb{P}(X=-4) = 2(\frac{1}{2}) + 3(\frac{1}{6}) - 4(\frac{1}{3}) = \frac{1}{6}$$



$$V(X) = \mathbb{E}[X^2] - \mathbb{E}^2[X] = [\sum_x x_i^2\,\mathbb{P}(X = x_i)] - \mathbb{E}^2[X] = [4(\frac{1}{2}) + 9(\frac{1}{6}) + 16(\frac{1}{3})] - \frac{1}{36} = 8.805$$

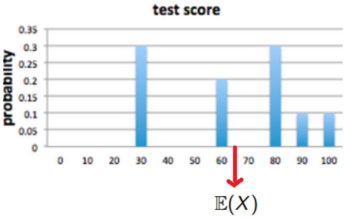
$$V(X) = \sum_i [x_i - \mathbb{E}[X]]^2\,\mathbb{P}(X = x_i) = (2 - \frac{1}{6})^2(\frac{1}{2}) + (3 - \frac{1}{6})^2(\frac{1}{6}) + (-4 - \frac{1}{6})^2(\frac{1}{3}) = 8.805$$

Random variables: Variance

- Example: Suppose that in a class of 10 people, grades on a test are given by 30, 30, 30, 60, 60, 80, 80, 80, 90, 100. Suppose a test is drawn from the pile at random and the score X is observed.

- Calculate the expected value of the randomly drawn test score.
- Calculate the variance according to this grading distribution.

x_i	30	60	80	90	100
$\mathbb{P}(X = x_i)$	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{1}{10}$



$$\mathbb{E}[X] = 30\,(\frac{3}{10}) + 60\,(\frac{2}{10}) + 80\,(\frac{3}{10}) + 90\,(\frac{1}{10}) + 100\,(\frac{1}{10}) = 64$$

$$V(X) = [30^2\,(\frac{3}{10}) + 60^2\,(\frac{2}{10}) + 80^2\,(\frac{3}{10}) + 90^2\,(\frac{1}{10}) + 100^2\,(\frac{1}{10})] - 64^2 = 624$$

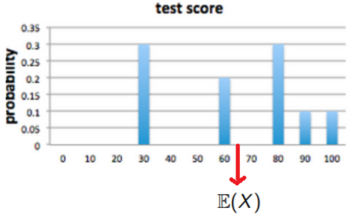
Random variables: Standard deviation

- Standard deviation: The standard deviation, denoted by σ, is the positive square root of the variance. It is the mean deviation of x_i from $\mathbb{E}$ and it is defined by:

$$\sigma(X) = \sqrt{V(X)}$$

- Example: The grading example

x_i	30	60	80	90	100
$\mathbb{P}(X = x_i)$	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{1}{10}$



$$\mathbb{E}[X] = 30\,(\frac{3}{10}) + 60\,(\frac{2}{10}) + 80\,(\frac{3}{10}) + 90\,(\frac{1}{10}) + 100\,(\frac{1}{10}) = 64$$

$$V(X) = [30^2\,\frac{3}{10} + 60^2\,\frac{2}{10} + 80^2\,\frac{3}{10} + 90^2\,\frac{1}{10} + 100^2\,\frac{1}{10}] - 64^2 = 624 \rightarrow \sigma = \sqrt{624} = 24.97$$

Random variables: Standard deviation

- Example: Consider the grading example and suppose now that grades are

x_i	30	60	70	80
$\mathbb{P}(X = x_i)$	$\frac{1}{10}$	$\frac{5}{10}$	$\frac{1}{10}$	$\frac{3}{10}$

$\mathbb{E}[X] = 30\,(\frac{1}{10}) + 60\,(\frac{5}{10}) + 70\,(\frac{1}{10}) + 80\,(\frac{3}{10}) = 64 \rightarrow$ The same expectation as the previous example

$V(X) = [30^2\ \frac{1}{10} + 60^2\ \frac{5}{10} + 70^2\ \frac{1}{10} + 80^2\ \frac{3}{10}] - 64^2 = 204 \rightarrow \sigma = \sqrt{204} = 14.28$

$\rightarrow$ The variance is smaller than the one of the previous example.

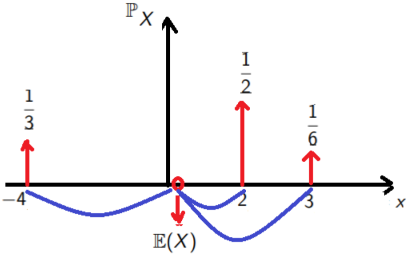
Random variables: Standard deviation

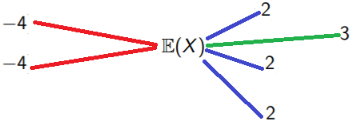
Example:

x_i	2	3	-4
$\mathbb{P}(X = x_i)$	$\mathbb{P}(X = 2) = \frac{1}{2}$	$\mathbb{P}(X = 3) = \frac{1}{6}$	$\mathbb{P}(X = -4) = \frac{1}{3}$

$$\mathbb{E}[X] = 2\ \mathbb{P}(X = 2) + 3\ \mathbb{P}(X = 3) + (-4)\ \mathbb{P}(X = -4) = 2(\frac{1}{2}) + 3(\frac{1}{6}) - 4(\frac{1}{3}) = \frac{1}{6}$$

$$V(X) \quad = \mathbb{E}[X^2] - \mathbb{E}^2[X] = [4\frac{1}{2} + 9\frac{1}{6} + 16\frac{1}{3}] - \frac{1}{36} = \frac{317}{36} \quad \to \quad \sigma(X) = \sqrt{\frac{317}{36}} = 2.96$$





Random variables: Variance

- Variance properties:
  - $V(aX) = a^2\ V(X)$

$$
\begin{aligned}
V(aX) &= \textstyle\sum_i [ax_i - \mathbb{E}[aX]]^2\ \mathbb{P}(X = x_i) \\
&= \textstyle\sum_i [ax_i - a\ \mathbb{E}[X]]^2\ \mathbb{P}(X = x_i) \\
&= \textstyle\sum_i a^2\ (x_i - \mathbb{E}[X])^2\ \mathbb{P}(X = x_i) \\
&= a^2\ \textstyle\sum_i (x_i - \mathbb{E}[X])^2\ \mathbb{P}(X = x_i) \\
&= a^2\ V(X)
\end{aligned}
$$

  - $V(c = constant) = 0$

$$
\begin{aligned}
V(c) &= \textstyle\sum_i (c - \mathbb{E}[c])^2\ \mathbb{P}(X = c) \\
&= \textstyle\sum_i (c - c)^2\ \mathbb{P}(X = c) \\
&= 0
\end{aligned}
$$

Random variables: Variance

- $V(a+X) = V(X)$

$$
\begin{aligned}
V(a+X) &= \sum_i [(a+x_i) - \mathbb{E}[a+X]]^2 \, \mathbb{P}(X = x_i) \\
&= \sum_i [a + x_i - (a + \mathbb{E}[X])]^2 \, \mathbb{P}(X = x_i) \\
&= \sum_i (a + x_i - a - \mathbb{E}[X])^2 \, \mathbb{P}(X = x_i) \\
&= \sum_i (x_i - \mathbb{E}[X])^2 \, \mathbb{P}(X = x_i) \\
&= V(X)
\end{aligned}
$$

Random variables: Variance

- $V(a - X) = V(X)$

$$
\begin{aligned}
V(a - X) &= \textstyle\sum_i [(a - x_i) - \mathbb{E}[a - X]]^2 \, \mathbb{P}(X = x_i) \\
&= \textstyle\sum_i [a - x_i - (a - \mathbb{E}[X])]^2 \, \mathbb{P}(X = x_i) \\
&= \textstyle\sum_i (a - x_i - a + \mathbb{E}[X])^2 \, \mathbb{P}(X = x_i) \\
&= \textstyle\sum_i (-x_i + \mathbb{E}[X])^2 \, \mathbb{P}(X = x_i) \\
&= \textstyle\sum_i (x_i - \mathbb{E}[X])^2 \, \mathbb{P}(X = x_i) \\
&= V(X)
\end{aligned}
$$

Random variables: Variance

- $V(aX+b) = a^2\ V(X)$

$$
\begin{aligned}
V(aX+b) &= \textstyle\sum_i [(ax_i+b) - \mathbb{E}[aX+b]^2\ \mathbb{P}(X=x_i) \\
&= \textstyle\sum_i [(ax_i+b) - (a\ \mathbb{E}[X]+b)]^2\ \mathbb{P}(X=x_i) \\
&= \textstyle\sum_i (ax_i+b-a\ \mathbb{E}[X]-b)^2\ \mathbb{P}(X=x_i) \\
&= \textstyle\sum_i (a\ x_i - a\ \mathbb{E}[X])^2\ \mathbb{P}(X=x_i) \\
&= a^2\ \textstyle\sum_i (x_i - \mathbb{E}[X])^2\ \mathbb{P}(X=x_i) \\
&= a^2\ V(X)
\end{aligned}
$$