Probability and Random Variables (ECE313/ECE317)

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Counting :
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Probability: Counting

- Principals of counting
  - Permutations
  - Arrangement
  - Combinations
  - Number of subsets
  - Partition

- Example (Basketball team):

A basketball coach has 20 players available. He need to choose 5 for the starting lineup and 7 who would be setting on the bench.

- Question:

In how many ways can the coach choose these 5 and 7 players?

Probability: Counting

Example1: A student has

- 4 shirts
- 3 pants
- 2 jackets

Number of possible combinations?

$4.3.2 = 24$

- The rule:

- n_i choices of stages $i \rightarrow \{4, 3, 2\}$
- Number of choices is: $n_1 \times n_2 \times n_3$

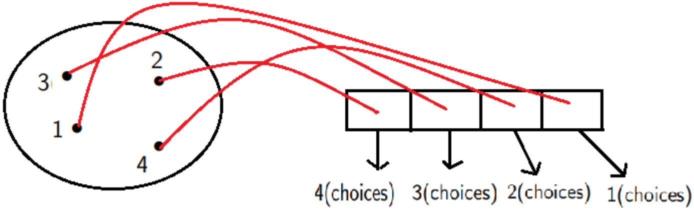
Probability: Counting

Example2:

- Number of licence plates with 2 letters followed by 3 digits:
  - How many different plates are there?
  - The answer is: $26 \,.\, 26 \,.\, 10 \,.\, 10 \,.10 = 676000$
  - How many different plates are there with no repeated letters and digits?
  - The answer is: $26 \,.\, 25 \,.\, 10 \,.\, 9 \,.8 = 468000$

Probability: Counting

- Permutations: Number of ways of ordering n elements



$$4.3.2.1 = 4! = 24$$

- The number of permutations of n elements $= n(n-1)(n-2)\dots 1 = n!$

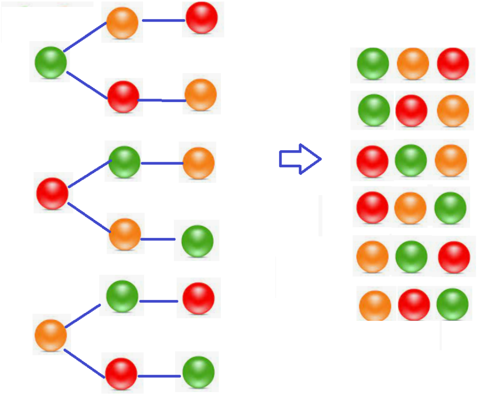
- Example:

How many words can we construct using the letters $\{$C, H, I, E, N$\}$

- Answer: $5! = 5.4.3.2.1 = 120$

Probability: Counting

- Permutation: Number of ways of ordering three balls {green, red, orange }



$$3.2.1 = 3! = 6 \ \rightarrow \ \text{The number of permutations}$$

Probability: Counting

Permutations with repetition:

Example1: 1 1 2 $\rightarrow$ The number of all permutations is $3! = 6$, and are given by:

112
112
– – – – – – 1 is permuted in 2! ways and gives identical elements 112
121
121
– – – – – – 1 is permuted in 2! ways and gives identical element 121
211
211
– – – – – – 1 is permuted in 2! ways and gives identical element 211

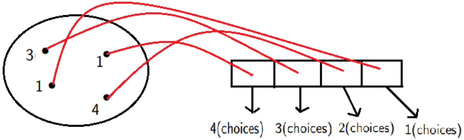
▷ Since each permutation of 1 will **give us the same number**, so the number of all these permutation is doubled $\rightarrow$ The correct number of permutations is $\frac{3!}{2!} = 3$

$$\{112, 121, 211\}$$

Probability: Counting

• Permutations with repetition: Number of ways of ordering n elements, but some elements are repeated

Example:



$$4.3.2.1 = 4! = 24$$

- But all the permutations of 1 will give us the same numbers $\rightarrow$ These numbers are repeated 2! times

$\Rightarrow$ Permutations of these four numbers with two times 1 are:

$$\frac{n!}{\#\text{permutations of } 1} = \frac{4!}{2!} = 12$$

Probability: Counting

Example2: C H E V R E E → The number of all permutations is 7!,

C H E V R E E
C H E V R E E
C H E V R E E
C H E V R E E
C H E V R E E
C H E V R E E
– – – – – – – – – – – – – – – – – – E is permuted in 3! ways

⋮

- Each permutation of "E" will give us the same word ⇒ All the words are repeated 3! times → The correct number of permutations is $\frac{7!}{3!} = \frac{7 \times 6 \times 5 \times 4 \times 3!}{3!} = 840$.

Probability: Counting

- STUDENT: $\rightarrow$ # permutations with repetition $= 7! = 5040$

$\Rightarrow$ $\{T\}$ is repeated two times: # permutations of $\{T\} = 2!$

$\Rightarrow$ Permutations of these seven letters with two times $\{T\}$ are: $\dfrac{7!}{2!} = 2520$

- SATANA: $\rightarrow$ # permutations with repetition $= 6! = 720$

$\Rightarrow$ $\{A\}$ is repeated three times: # permutations of $\{A\} = 3!$

$\Rightarrow$ Permutations of these six letters with three times $\{A\}$ are: $\dfrac{6!}{3!} = 120$

- BANANA: $\rightarrow$ # permutations with repetition $= 6! = 720$

$\Rightarrow$ $\{A\}$ is repeated three times: # permutations of $\{A\} = 3!$

$\Rightarrow$ $\{N\}$ is repeated two times: # permutations of $\{N\} = 2!$

$\Rightarrow$ Permutations of these six letters with 3 times $\{A\}$ and 2 times $\{N\}$ are: $\dfrac{6!}{3!2!} = 60$

- MAMA: $\rightarrow$ # permutations with repetition $= 4! = 24$

$\Rightarrow$ $\{A\}$ is repeated two times: # permutations of $\{A\} = 2!$

$\Rightarrow$ $\{M\}$ is repeated two times: # permutations of $\{M\} = 2!$

$\Rightarrow$ Permutations of these four letters with 2 times $\{A\}$ and 2 times $\{M\}$ are: $\dfrac{4!}{2!2!} = 6$

Probability: Counting

Example

Experiment: Six rolls of a (six-sided) die $\rightarrow$ $|\Omega| = 6^6 = 46656$

- Typical outcome : $\{2, 3, 5, 6, 3, 2\}$

- Event A: $\{$All rolls give different numbers$\}$ $->$ $(2, 3, 5, 6, 4, 1)$

$$|A| = \#\text{permutations} = 6! = 720 \Rightarrow \mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{6!}{6^6} = 0.0154 = 1.54\%$$

-Event B: $\{$ Only 5 is repeated two times$\}$ $->$ $(2, 3, 5, 6, 5, 1)$

$->$ We should compute all the permutations of the following cases:

$$\{1, 2, 3, 4, 5, 5\}, \{1, 2, 3, 5, 5, 6\}, \{1, 2, 5, 5, 4, 6\}, \{1, 5, 5, 3, 4, 6\}, \{5, 5, 2, 3, 4, 6\}$$

$$\Rightarrow |B| = 5 \times \frac{6!}{2!} = 1800 \;\Rightarrow \mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{1800}{6^6} = 0.00385 = 0.38\%$$

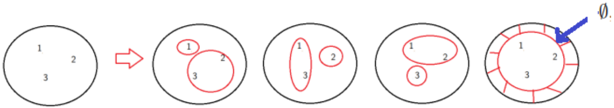
Probability: Counting

- Number of subsets:

Consider a set which consist of n elements: $\{1, 2, \ldots, n\}$

- Question:

How many subsets can we construct from that set?



$\Rightarrow$ Each element has two choices: **Be** in the subset or **Not**.

$$\{1, 2, \ldots, n\} \rightarrow 2.2.2\ldots2 = 2^n \Rightarrow \text{The number of subsets} = 2^n$$

- $n = 3 \Rightarrow$ The number of subsets

$2^3 = 8 = \{\{1\}, \{2, 3\}, \{2\}, \{1, 3\}, \{3\}, \{1, 2\}, \emptyset, \{1, 2, 3\}\}$

- $n = 4 \Rightarrow$ The number of subsets

$2^4 = 16 = \{\{1\}, \{2, 3, 4\}, \{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}, \{1, 4\}, \{2, 3\}, \{1, 2, 3, 4\}, \emptyset$
$\{2\}, \{1, 3, 4\}, \{3\}, \{1, 2, 4\}, \{4\}, \{1, 2, 3\}\}$

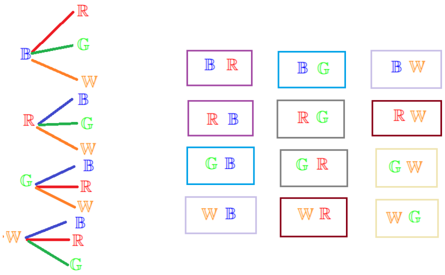
Probability: Counting

- Arrangement: **An arrangement** of k distinct elements from a set of n elements, denoted $\mathbb{A}_k^n$, $\rightarrow$ is an ordered sequences of these k elements $\rightarrow$ **The order matters**.

Example: An urn contains 4 balls of different colors. I want to withdraw 2 balls. How many possibilities do I have?

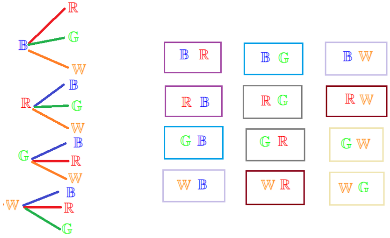
- Method 1: Withdraw the first ball from the urn **without return it back** $\rightarrow$ see how many possibilities do I have. Than withdraw the second ball.

$\rightarrow$ The number of all possible combinations is $4 \times 3 = 12$, as presented in the following graph.



Probability: Counting

- We can remark that **the order matters** and the following results are all repeated two times: $\{\mathbb{BR}, \mathbb{BG}, \mathbb{BW}, \mathbb{RG}, \mathbb{GW}, \mathbb{RW}\} \rightarrow \{\mathbb{BR}$ and $\mathbb{RB}$ are **two different elements**$\}$



$$\boxed{A_k^n = n(n-1)(n-2)\dots(n-k+1) = \frac{n!}{(n-k)!}} \rightarrow \text{(Arrangement)}$$

$$\mathbb{A}_2^4 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{4 \times 3 \times 2!}{2!} = 4 \times 3 = 12$$

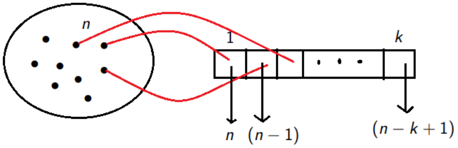
$\boxed{\mathbb{A}_n^n = n!}$ $\rightarrow$ A Permutation is an arrangement of n elements from a set of n elements

Probability: Counting

- Combination: We want to take a combination of k elements of the original set of n different elements **without replacement** $\rightarrow$ **The order doesn't matter** $=$ **The number of subsets containing k elements**.

- Question: How it can be done?

- Method 1: Choose the k elements one by one without replacement



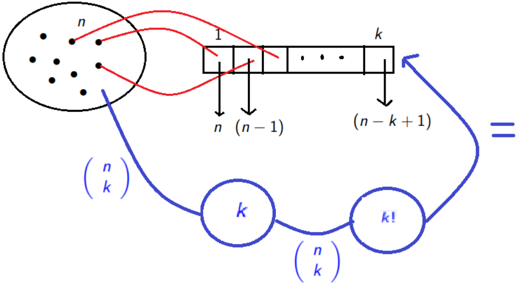
$$n(n-1)(n-2)\dots(n-k+1) = \frac{n!}{(n-k)!} = \mathbb{A}_k^n \rightarrow \text{(Arrangement)}$$

$\rightarrow$ But we will get a repeated subsets since using this method, the order matter, and each subset is repeated $k!$ times $\Rightarrow$ $C_k^n = \dfrac{\mathbb{A}_k^n}{k!} = \dfrac{n!}{k!(n-k)!}$

- Notation: $C_k^n = \begin{pmatrix} n \\ k \end{pmatrix}$ = number of subsets of k elements from a given n elements

Probability: Counting

- Method 2: Choose the k elements at ones than ordered them (do the permutation of all these k elements) $\rightarrow$ **We will get all the possible arrangements**



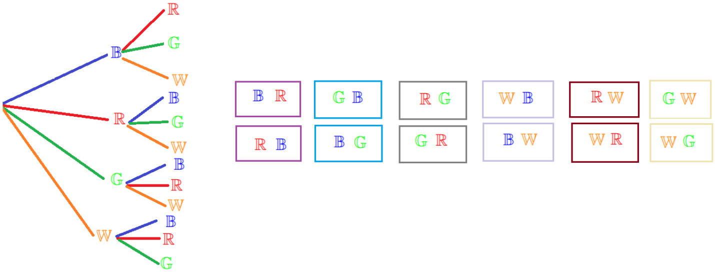
$$\underbrace{k! + k! + k! + \ldots + k!}_{\text{number of subsets of k elements}} = \#\text{subsets} \times k! = \binom{n}{k} k! = \mathbb{A}_k^n$$

$$\Rightarrow \binom{n}{k} k! = n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!} \Rightarrow \boxed{\binom{n}{k} = C_k^n = \frac{n!}{k!(n-k)!}}$$

Probability: Combination

Example: An urn contains 4 balls of different colors. How many subsets of 2 elements can we get $\rightarrow$ **The order doesn't matter**?

- All the possible combinations are 12 and presented as follow

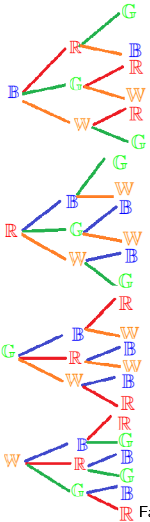


$n = 4,\quad k = 2 \;\Rightarrow\; C_2^4 \times 2! = \mathbb{A}_2^4 = \dfrac{4!}{(4-2)!} = 12 \;\rightarrow$ **But each two elements are repeated 2! times** $\;\Rightarrow C_2^4 = \dfrac{\mathbb{A}_2^4}{2!} = \dfrac{12}{2!} = 6 \Rightarrow C_k^n = \dfrac{n!}{k!(n-k)!}$

- The number of subsets is 6: $\{\mathbb{BR},\ \mathbb{BG},\ \mathbb{BW},\ \mathbb{RG},\ \mathbb{GW},\ \mathbb{RW}\}$

Probability: Combination

Example: An urn contains 4 balls of different colors. How many subsets of 3 elements can we get $\rightarrow$ **The order doesn't matter**?



Probability: Combination

- All the possible arrangements are $\mathbb{A}_3^4 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 24$

$n = 4, \quad k = 3 \;\Rightarrow\; C_3^4 \times 3! = \mathbb{A}_3^4 = \frac{4!}{(4-3)!} = 24 \;\rightarrow$ **But each two elements are repeated 3! times** $\Rightarrow C_k^n = \frac{\mathbb{A}_3^4}{3!} = \frac{24}{3!} = 4$

- The number of subsets is 4: $\{\mathbb{BRG},\ \mathbb{BGW},\ \mathbb{BWR},\ \mathbb{RGW}\}$

- The basket ball team example: In how many ways a basketball coach can choose 5 players for the starting lineup and 7 players who would be setting on the bench

$$C_5^{20} = \frac{20!}{5!(20-5)!} = \frac{20!}{5!15!} = 15504,$$

$$C_7^{15} = \frac{15!}{7!(15-7)!} = \frac{15!}{7!8!} = 6435.$$

Probability: Counting

- Properties of combination:

- $0! = 1$ (by convention)
- $C_n^n = \dfrac{n!}{n!(n-n)!} = \dfrac{n!}{n!0!} = \dfrac{n!}{n!} = 1;$
- $C_0^n = \dfrac{n!}{0!(n-0)!} = \dfrac{n!}{n!} = 1 \rightarrow$ one set which is the empty set $\rightarrow \{\emptyset\}$
- $\sum_{k=0}^{n} C_k^n = C_0^n + C_1^n + \dots C_n^n = 2^n \rightarrow$ number of all subsets = sum of all combinations.

Remark: We can proof the above property using the Newton (or binomial) formula:

$$(a+b)^n = \sum_{k=0}^{n} C_k^n \; a^k \; b^{n-k}$$

$$\text{For } a = 1 \text{ and } b = 1 \Rightarrow 2^n = (1+1)^n = \sum_{k=0}^{n} C_k^n \; 1^k \; 1^{n-k} = \sum_{k=0}^{n} C_k^n$$

Probability: Arrangement

- An arrangement with repetition: An arrangement of k elements from a set of n elements, $\rightarrow$ **but we have right to take the same element many times.**

Example:

An urn contains 4 balls of different colors. We want to take two elements in the following way: Withdraw the first ball from the urn and see how many possibilities I have $\rightarrow$ **Return it back** than withdraw the second ball.

- All the possible combinations are: $4 \times 4 = 4^2 = 16$

$$\{\mathbb{BR}, \mathbb{RB}, \mathbb{BG}, \mathbb{GB}, \mathbb{BW}, \mathbb{WB}, \mathbb{RG}, \mathbb{GR}, \mathbb{WR}, \mathbb{RW}, \mathbb{GW}, \mathbb{WG}, \mathbb{BB}, \mathbb{RR}, \mathbb{GG}, \mathbb{WW}\}$$

$$\mathrm{R}_k^n = \underbrace{n \times n \times n \times \ldots \times n}_{k \text{ times}} = n^k \rightarrow \text{Arrangement with repetition}$$

Example:

rolling a die 4 times, is an arrangement with repetition:

$$\mathbb{R}_4^6 = 6 \times 6 \times 6 \times 6 = 6^4 = 1296 \rightarrow \text{possible cases}$$

Probability: Combination

- Combination with repetition: How many ways can k element be selected form a collection of n objects where **the order doesn't matter (subsets)** and **repetition allowed.**

Example:

An urn contains 4 balls of different colors. We want to take two elements such that **the order doesn't matter** and **the repetition is allowed:**

- All the possible combinations are:

$$\{\mathbb{BR},\ \mathbb{BG},\ \mathbb{BW},\ \mathbb{RG},\ \mathbb{WR},\ \mathbb{WG},\ \mathbb{BB},\ \mathbb{RR},\ \mathbb{GG},\ \mathbb{WW}\} = 10 \text{ elements}$$

- The rule:

$$\boxed{K_k^n = C_k^{n+k-1} = \frac{(n+k-1)!}{k!(n-1)!} = C_{n-1}^{n+k-1} \ \rightarrow \ \text{Combination with repetition}}$$

$$\mathbb{K}_2^4 = C_2^{2+4-1} = C_2^5 = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{5 \times 4 \times 3!}{2!3!} = 10$$

Probability: Counting (The order and Repetition)

$$\underbrace{\text{Take } k \text{ elements from } n \text{ elements}}$$

$$\Downarrow$$

- The order matters + without repetition $\rightarrow$ $\mathbb{A}_k^n = \frac{n!}{(n-k)!}$
- The order matters + with repetition $\rightarrow$ $\mathbb{R}_k^n = n^k$
- The order doesn't matter + without repetition $\rightarrow$ $C_k^n = \dfrac{n!}{k!(n-k)!}$
- The order doesn't matter + with repetition $\rightarrow$ $\mathbb{K}_k^n = C_k^{n+k-1} = \dfrac{(n+k-1)!}{k!(n-1)!}$

$$\underbrace{\text{Take } n \text{ elements from } n \text{ elements}}$$

- The order matters + without repetition $\rightarrow$ $\mathbb{A}_n^n = n!$ (permutation)
- The order matters + with repetition $\rightarrow$ $\mathbb{R}_n^n = n^n$
- The order doesn't matter + without repetition $\rightarrow$ $C_n^n = 1$
- The order doesn't matter + with repetition $\rightarrow$ $\mathbb{K}_n = C_n^{2n-1} = \dfrac{(2n-1)!}{n!(n-1)!}$

Probability: Counting

- Remark: In counting we should take into consideration if:

The order matters or not and if the repetition is allowed or not

- Example1: how many words of 3 letters can be made?

1) With repetition + the order matters: $\mathbb{R}_3^{26} = 26 \times 26 \times 26 = 26^3 = 17576$

2) Without repetition + the order matters:

$$\mathbb{A}_3^{26} = 26 \times 25 \times 24 = \frac{26!}{(26-3)!} = \frac{26!}{23!} = 15600$$

3) Without repetition + the order doesn't matter: $C_3^{26} = \frac{26!}{3!(26-3)!} = \frac{26!}{3!23!} = 2600$

4)The repetition is allowed + the order doesn't matter:

$$\mathbb{K}_3^{26} = C_3^{26+3-1} = C_3^{28} = \frac{28!}{3!(28-3)!} = \frac{28!}{3!25!} = 3276$$

Probability: Counting

- Example2: Consider the US phone numbers starting by 865

$$865 \quad * \ * \ * \ * \ * \ * \ *$$

1) How may phone number can we get?
$\Rightarrow$ It is an arrangement with repetition of 7 numbers form $\{0,1,2,3,4,5,6,7,8,9\}$

$$\mathbb{R}_7^{10} = 10^7 = 10.000.000$$

2) In the US we are more than 300 millions person. How can we get phone numbers for all the population? $\Rightarrow$ We can construct 3 digits area codes from

- Case1 An arrangement with repetition of 3 numbers from $\{0,1,2,3,4,5,6,7,8,9\}$

$$\mathbb{R}_3^{10} = 10^3 = 1000$$

- Case2 An arrangement without repetition of 3 numbers from $\{0,1,2,3,4,5,6,7,8,9\}$

$$\mathbb{A}_3^{10} = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 720 \text{ possibilities}$$

$\Rightarrow$ We can give to each state one or more different 3 digits area codes from the 720 numbers.

Probability: Distribution

- Example1: Let the set $\Omega = \{1, 2, 3\}$. We want to give to Alice 1 element and to Ben 2 elements. In how many ways it can be done?

- Let us consider all the permutations of these three elements $= 3! = 6$

1 2 3
1 3 2
$- - - - - - - - - - - - - - - - - - = \; 1!\,2!$ permutations (sets)
2 1 3
2 3 1
$- - - - - - - - - - - - - - - - - - = \; 1!\,2!$ permutations (sets)
3 1 2
3 2 1
$- - - - - - - - - - - - - - - - - - = \; 1!\,2!$ permutation (sets)
$= n! = 6$ All the permutations

$$C\;1!\;2! = 3! \rightarrow (C = \#\text{distribution}) \Rightarrow C = \frac{3!}{1!\;2!} = 3 = (\{1\},\{2,3\}),(\{2\},\{1,3\}),(\{3\},\{1,2\}).$$

Probability: Partition (Distribution)

- Example1: Let the set $\Omega = \{1, 2, 3, 4\}$. We want to give to Alice 1 element and to Ben 3 elements. In how many ways it can be done?
- Let us consider all the permutations of these four elements $= 4! = 24$

1 2 3 4	2 1 3 4	3 2 1 4	4 1 3 2
1 2 4 3	2 1 4 3	3 2 4 1	4 1 2 3
1 3 2 4	2 3 1 4	3 1 2 4	4 3 1 2
1 3 4 2	2 3 4 1	3 1 4 2	4 3 2 1
1 4 2 3	2 4 1 3	3 4 2 1	4 2 1 3
1 4 3 2	2 4 3 1	3 4 1 2	4 2 3 1
$= 1!\ 3!$	$= 1!\ 3!$	$= 1!\ 3!$	$= 1!\ 3!$
one distribution	one distribution	one distribution	one distribution

$$C\ 1!\ 3! = 4! \to (C = \#\text{distribution}) \Rightarrow C = \frac{4!}{1!\ 3!} = 4$$

Probability: Distribution

- Example2: Let the set $\Omega = \{1, 2, 3, 4\}$. We want to give to Alice 2 elements and to Ben 2 elements. In how many ways it can be done? $\Rightarrow$ All the permutations are $= 4! = 24$

1 2 3 4	1 3 2 4	1 4 2 3	2 3 1 4	2 4 1 3	3 4 1 2
1 2 4 3	1 3 4 2	1 4 3 2	2 3 4 1	2 4 3 1	3 4 2 1
2 1 3 4	3 1 2 4	4 1 2 3	3 2 1 4	4 2 1 3	4 3 1 2
2 1 4 3	3 1 4 2	4 1 3 2	3 2 4 1	4 2 3 1	4 3 2 1
$= 2!\,2!$	$= 2!\,2!$	$= 2!\,2!$	$= 2!\,2!$	$= 2!\,2!$	$= 2!\,2!$

Alice	Ben
$\{1, 2\}$	$\{3, 4\}$
$\{1, 3\}$	$\{2, 4\}$
$\{1, 4\}$	$\{2, 3\}$
$\{3, 4\}$	$\{1, 2\}$
$\{2, 4\}$	$\{1, 3\}$
$\{2, 3\}$	$\{1, 4\}$

$\rightarrow C = \#\text{distributions} \Rightarrow C = \dfrac{4!}{2!\,2!} = 6$

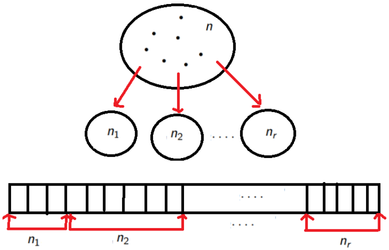
Probability: Distribution

- Let a set of $n \geq 1$ elements. I have r persons. I want to give n_i elements to each person i such that $n_1 + n_2 + n_3 + \ldots + n_r = n$

- Question: In how many ways can I distribute these n_i elements to these r persons?

- Denote by C = the number of possible distributions

$\Rightarrow$ We give n_i items to person i $\Rightarrow$ We compute the permutations of each n_i $\rightarrow$ $= n_i!$



$C\ n_1!\ n_2!\ n_3! \ldots n_r! = n! \rightarrow (C = \#\text{distributions}) \Rightarrow$ $$\#\ \text{distributions} = C = \frac{n!}{n_1!\ n_2!\ n_3! \ldots n_r!}$$

Probability: Distribution

- Another way of thinking:

1) I choose n_1 elements from the $n = n_1 + n_2 + \ldots + n_r$ elements $\rightarrow C_{n_1}^{n} = \dfrac{n!}{n_1!(n-n_1)!}$

2) I choose n_2 elements from the $n - n_1$ elements $\rightarrow C_{n_2}^{n-n_1} = \dfrac{(n-n_1)!}{n_2!(n-n_1-n_2)!}$

3) I choose n_3 elements from the $n - n_1 - n_2$ elements $\rightarrow C_{n_3}^{n-n_1-n_2} = \frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!}$

$\vdots$

r) I choose n_r elements from the $n - n_1 - n_2 - \ldots - n_{r+1}$ elements $\rightarrow$
$C_{n_r}^{n-n_1-n_2-\ldots-n_{r+1}} = \frac{(n-n_1-n_2-\ldots-n_{r+1})!}{n_r!(n-n_1-n_2-\ldots-n_{r+1}-n_r)!}$

$$C = C_{n_1}^{n} \times C_{n_2}^{n-n_1} \times C_{n_3}^{n-n_1-n_2} \times \ldots C_{n_r}^{n-n_1-n_2-\ldots-n_{r+1}}$$

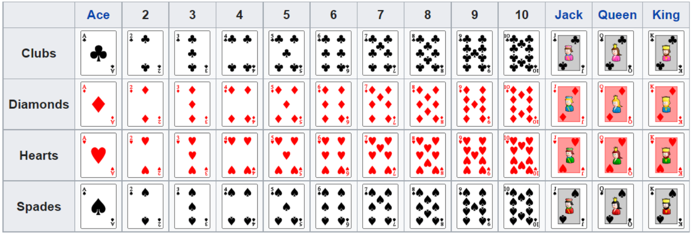
$$= \frac{n!}{n_1!(n-n_1)!} \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!} \ldots \frac{(n-n_1-n_2-\ldots-n_{r+1})!}{n_r!(n-n_1-n_2-\ldots-n_{r+1}-n_r)!}$$

$$= \frac{n!}{n_1!\ n_2!\ n_3!\ldots n_r!}$$

Probability: Distribution

Example

- 52 card deck dealt fairly to 4 players.



Find $\mathbb{P}$(each player gets an Ace)?

- **Outcomes Ω:** Partition of 52 cards into 4 subsets

$$n_i = 13, \quad i = 1, 2, 3, 4 \;\Rightarrow\; \#\text{distributions} = \frac{52!}{13!\;13!\;13!\;13!} = |\Omega|$$

Probability: Partition (Distribution)

- Event A: Each player gets an Ace

- Constructing outcomes with one Ace for each player:

- Distribute the Aces: $4 \times 3 \times 2 \times 1 = 24$

$\Rightarrow$ distribution of 4 cards on 4 persons $= \dfrac{4!}{1!\ 1!\ 1!\ 1!} = 4! = 24$

- Distribute the remaining 48 cards: $\boxed{\text{Ace}}\ \underbrace{\bullet\ \bullet\ \bullet\ \ldots\ \bullet}_{48\text{cards}} = \dfrac{48!}{12!\ 12!\ 12!\ 12!}$ -

Each player gets an Ace $\Rightarrow$ $|A| = 24 \times \dfrac{48!}{12!\ 12!\ 12!\ 12!}$

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = 24 \times \frac{\dfrac{48!}{12!\ 12!\ 12!\ 12!}}{\dfrac{52!}{13!\ 13!\ 13!\ 13!}} = 0.1055.$$

Probability: Binomial Probability

- Suppose that we are tossing a fair coin six times, such that $\mathbb{P}(H) = p$

$\rightarrow$ The number of possible cases is 2^6 $\rightarrow$ $|\Omega| = 2^6 = 64$

- Question: What is the probability of getting 4 Heads? $\rightarrow$ $\mathbb{P}$(4 Heads).

- Event A: Getting 4 Heads $\rightarrow$ {(HTTHHH), (HTHHHH), ...}

$$|A| = C_4^6 = \frac{6!}{4!(6-4)!} = \frac{6!}{4!2!} = 15 \quad \rightarrow \text{(15 outcomes with 4 Heads)}$$

- $\{(HTTHHH), (HTHHHH), ...\}$ can be seen as a permutation with repetition $\frac{6!}{4!2!} = 15$.

$$\mathbb{P}(HTTHHH) = p(1-p)(1-p)ppp = p^4(1-p)^2 = p^{\#Heads}(1-p)^{\#Tails}$$

$\Rightarrow$ **The probability of a typical outcome containing 4 Heads**

- **We have 15 outcomes** $\Rightarrow \mathbb{P}(A) = \mathbb{P}(4\ Heads) = C_4^6 p^4(1-p)^2 = 15p^4(1-p)^2$

$\Rightarrow$ The sum of probabilities of all the sequences of 4 Heads

Probability: Binomial Probability

- In particular, if

$$\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$$

$$\mathbb{P}(A) = \mathbb{P}(4\ Heads) = C_4^6 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2 = 15 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2 = 15 \times 0.0625 \times 0.25 = 0.2343$$

Remark1:
Since in this case, Head and Tail are equally likely, we can use the discrete probability law to compute the probability of the event A

$$\mathbb{P}(4\ Heads) = \frac{|A|}{|\Omega|} = \frac{15}{64} = 0.2343$$

Remark2:
The number of outcomes of 4 Heads = The number of outcomes of 2 Tails

$$\#\text{outcomes with } 4\ Heads = C_4^6 = \frac{6!}{4!\ 2!} = \#\text{outcomes with } 2\ Tails = C_4^6 = \frac{6!}{2!\ 4!}$$

Probability: Binomial Probability

- Suppose that we are tossing a fair coin six times, such that $\mathbb{P}(H) = p$

$\rightarrow$ The number of possible cases is 2^6 $\rightarrow$ $|\Omega| = 2^6 = 64$

- Question: What is the probability of getting 1 Tail ?

- Event B: Getting 1 Tail $\rightarrow \{(HTHHHH), (HHHTHH), ...\}$

$$|B| = C_1^6 = \frac{6!}{1!(6-1)!} = \frac{6!}{1!5!} = 6 \quad \rightarrow \text{(6 outcomes with 1 Tail)}$$

$$\mathbb{P}(HTHHHH) = p^5(1-p) \;\rightarrow\; \text{The probability of a typical outcome with 1 Tail}$$

$$\mathbb{P}(B) = \mathbb{P}(1\ Tail) = C_1^6 p^5(1-p) = 6p^5(1-p)$$

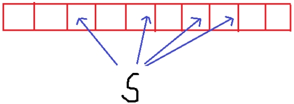
- If $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2} \Rightarrow \mathbb{P}(B) = 6\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right) = 0.09375 = \frac{|B|}{|\Omega|} = \frac{6}{64} = 0.09375$

Probability: Binomial Probability

Definition:

A binomial law is the repetition of n identical and independent experiments with two possible outcomes that we can call **"success"** and **"failure"**

- Event A: Getting k **"successes"** $\Rightarrow$ getting $n-k$ **"failures"**



$|A| =$ In how many ways can I place these k items in the n boxes $= C_k^n = \dfrac{n!}{k!(n-k)!}$.

$\rightarrow$ It can be seen as a permutation with repetition of k successes and (n-k) failures

$\Rightarrow$ The probability of each element of A is $p^k(1-p)^{n-k}$

$$\mathbb{P}(A) = C_k^n \ p^k(1-p)^{n-k} = C_k^n \ p^{\#\ successes}(1-p)^{\# failures}$$

$C_k^n \ p^k(1-p)^{n-k} \rightarrow$ **binomial probability**

Probability: Binomial Probability

Example

- Given that there are 3 Heads in 10 tosses of a coin, such that $\mathbb{P}(H) = p$. What is the probability that the first two tosses were Heads?
- Event A: There are 3 Heads in 10 tosses.

$$\mathbb{P}(A) = C_3^{10} p^3 (1-p)^7 = 120\ p^3 (1-p)^7$$

- Event B: The first two tosses are Heads.

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(HH \text{ and one H in 8 tosses})}{\mathbb{P}(A)} = \frac{p^2 \left[C_1^8\ p(1-p)^7\right]}{C_3^{10}\ p^3(1-p)^7} = \frac{C_1^8}{C_3^{10}} = \frac{1}{15}$$

Examples- Training

Binomial	$\{1,2,3\dots,n\}$
$- - - - - --$	$- - - - - --$
$S \qquad F$	n elements
$- - - - - --$	$- - - - - --$
$\Downarrow$	$\Downarrow$
repeat the experiment n times	$\{1,2,3\dots,n\}$
$\Downarrow$	$\Downarrow$
All possible cases $= 2^n$	$\{1,2,3\dots,n\}$
$\Downarrow$	$\Downarrow$
typical outcome $= (\underbrace{SFSSF\dots}_{n \text{ items}})$	$\{1,2,3\dots,n\}$

Remark: In the binomial law the possible cases of k successes from n experiments can be seen as a combination of k elements from n or as a permutation with repetition or as a distribution of n elements within two person by giving k elements to one of them and $n-k$ to the second one

$$\frac{n!}{k!(n-k)!}$$

Examples- Training

Example1:

A student have right to repeat the Exam three times. The probability that this student will success is $\mathbb{P}(S) = \frac{2}{3}$, and the probability that he will fail is $\mathbb{P}(F) = \frac{1}{3}$

a) What is the probability that this student will fail 2 times?

b) What is the probability that this student will success 2 times

c) What is the probability that this student will not fail?

Examples- Training

Example1-Solution:

- All the possibile cases are:

$$\{SSS, SSF, SFS, SFF, FSS, FSF, FFS, FFF\} = 2^3 = 8$$

a) Event A:{The student will fail 2 times}

$\rightarrow$ All the possibilities are: $\{FSF, FFS, SFF\} = C_2^3 = \frac{3!}{2!1!} = 3$

$\rightarrow$ The probability of each outcome is: $= \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)$

$$\Rightarrow \mathbb{P}(A) = C_2^3 \times \left[\left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)\right] = 3\left[\left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)\right] = \frac{2}{9} = 0.222$$

Examples- Training

Example1-Solution:

- All the possibile cases are:

$$\{SSS, SSF, SFS, SFF, FSS, FSF, FFS, FFF\} \ = 2^3 = 8$$

b) Event B:{The student will success 2 times}

$\rightarrow$ All the possibilities are:$\{SSF, FSS, SFS\} \ = C_2^3 = \dfrac{3!}{2!1!} = 3$

$\rightarrow$ The probability of each outcome is: $= \left(\dfrac{1}{3}\right) \left(\dfrac{2}{3}\right)^2$

$$\Rightarrow \mathbb{P}(B) = C_2^3 \times \left[\left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^2\right] = 3\left[\left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^2\right] = \frac{4}{9} = 0.444$$

Examples- Training

Example1-Solution:

- All the possibile cases are:

$$\{SSS, SSF, SFS, SFF, FSS, FSF, FFS, FFF\} = 2^3 = 8$$

c) Event C:{The student will not fails}

$\rightarrow$ All the possibilities are:$\{SSS\} = C_3^3 = \frac{3!}{3!} = 1$

$\rightarrow$ The probability of each outcome is: $= \left(\frac{2}{3}\right)^3$

$$\Rightarrow \mathbb{P}(C) = C_3^3 \times \left(\frac{2}{3}\right)^3 = 1 \times \left(\frac{2}{3}\right)^3 = \frac{8}{27} = 0.296$$

Examples- Training

Example2:

An airline has set up an overbooking system for one of its lines in order to lower costs. Reservations can only be made through an agency or on the company's website. On this line, the company charters an aircraft of 200 seats and has sold 202 reservations. It is assumed that the probability of a customer to arrive for boarding is $p = 0.971$

1- Calculate the probability that all customers will show up for boarding.

2- Calculate the probability that a single customer among those who booked does not show up for boarding.

3- Deduce the probability that the company is in an overbooked situation (i.e. with more customers arriving for boarding than seats).

Examples- Training

Example2-Solution:

Since the probability that the customer will show up (success) is $p = 0.971 \Rightarrow$ the probability that the customer will not show up (failure) is $1 - p = 0.029$.

1- Event A: {All customers will show up for boarding.}

$$\mathbb{P}(A) = C_{202}^{202} \times p^{202} \times (1-p)^0 = 1 \times (0.971)^{202} \times 1 = 0.0026$$

$$\begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & \dots & C_{202} \\ \downarrow & \downarrow & \downarrow & \downarrow & \dots & \downarrow \\ 0.971 & 0.971 & 0.971 & 0.971 & \dots & 0.971 \end{array} \rightarrow p = 0.971^{202} = 0.0026$$

2- Event B: {A single customer among those who booked will not show up for boarding}

$$\mathbb{P}(B) = C_{201}^{202} \times p^{201} \times (1-p)^1 = 202 \times (0.971)^{201} \times 0.029 = 0.015.$$

3- Event C:{The company is in an overbooked situation}

$\Rightarrow$ {All customers will show up (202 person)} OR {just one will not show up (201 person)}

$\Rightarrow$ $C = A \cup B$ $\rightarrow$ A and B are disjoints (cannot happen at the same time)

$$\mathbb{P}(C) = \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) = 0.0026 + 0.015 = 0.0176$$

Examples- Training

Example3:

How many different 6-digit numbers are there

a) If there are no restrictions?

b) If the numbers must be divisible by 5?
(A number is divisible by 5 if the last digit is either 0 or 5.)

c) when the repetition of digits is excluded?

Examples- Training

Example3-Solution:

a) It is an arrangement with repetitions.

$\rightarrow$ The first digit cannot be 0 because if it were, the number would have 5 digits.

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 9 & 10 & 10 & 10 & 10 & 10 \end{array} \quad = 9 \times \mathbb{R}_5^{10} = 9 \times 10^5 = 900000$$

b) If the numbers must be divisible by 5?

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 9 & 10 & 10 & 10 & 10 & \{0,5\} \end{array} \quad = 9 \times \mathbb{R}_4^{10} \times C_1^2 = 9 \times 10^4 \times 2 = 180000$$

c) The repetition of digits is excluded $\rightarrow$ It is an arrangement without repetitions

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 9 & 9 & 8 & 7 & 6 & 5 \end{array} \quad = 9 \times \mathbb{A}_5^9 = 9 \times \frac{9!}{(9-5)!} = 9 \times \frac{9!}{4!} = 136080$$

Examples- Training

Example4: In a group there are 10 men, 8 women and 7 children. In how many different ways can we place them on a line if

a) They can position themselves freely?

b) Men want to stay together and there order doesn't matter?

C) Men want to stay together and there order matter?

Example4-Solution:

a) It is a permutation without repetitions $= (10 + 8 + 7)! = 25!$

b) Men can be considered as one item $\Rightarrow (\underbrace{1}_{\text{group of men}} +8 + 7)! = 16!$

c) When men are considered as one item, the possible orders are 16!. In each order men have 10! possible position in there group $\Rightarrow$ All possible orders are

$$10! \times 16!$$

Examples- Training

Example5:

A three-wheeled number padlock; each can be a number from 0 to 9. How many secret "numbers" are there?

Example5-Solution:

It is an arrangement with repetitions $= \mathbb{R}_3^{10} = 10^3 = 1000$

- A padlock with more wheeled numbers is more safe.

Examples- Training

Example6:

In an urn, there are 5 red balls (R), 2 blue balls (B) and 1 green ball (G), indistinguishable by touch. Two balls are drawn successively and without replacement(we will not return back the ball). We want to determine the probability of drawing two balls of the same color.

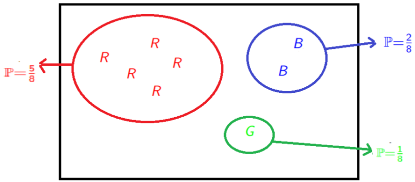
1. Represent on a tree all the possibilities by indicating on the corresponding branches the probability of getting two balls from each draw.

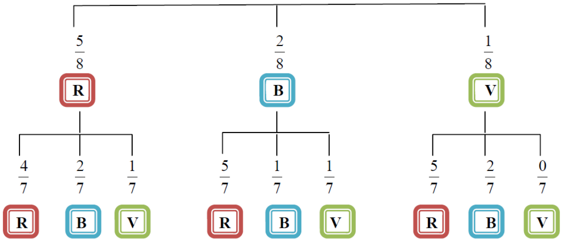
2. Deduce the probability of having: the couple (R, R), the couple (B, B) and the couple (G, G).

3. Deduce the probability of drawing two balls of the same color.

Examples- Training

Example6-Solution: $\underbrace{R\ R\ R\ R\ R}_{\mathbb{P}(R)=\frac{5}{8}}\ \ \underbrace{B\ B}_{\mathbb{P}(B)=\frac{2}{8}}\ \ \underbrace{G}_{\mathbb{P}(G)=\frac{1}{8}}$





Examples- Training

Example6-Solution:

$$\underbrace{R\ R\ R\ R\ R}_{\mathbb{P}(R)=\frac{5}{8}}\quad \underbrace{B\ B}_{\mathbb{P}(B)=\frac{2}{8}}\quad \underbrace{G}_{\mathbb{P}(G)=\frac{1}{8}}$$

$$\mathbb{P}(R,R) = \frac{5}{8} \times \frac{4}{7} = \frac{20}{56} \rightarrow \text{Since the ball is not replaced in the urn}$$

$$\mathbb{P}(B,B) = \frac{2}{8} \times \frac{1}{7} = \frac{2}{56} \rightarrow \text{Since the ball is not replaced in the urn}$$

$$\mathbb{P}(G,G) = \frac{1}{8} \times \frac{0}{7} = \frac{0}{56} = 0 \rightarrow \text{Since the ball is not replaced in the urn}$$

- The probability of getting two balls of the same color

$$= \mathbb{P}(R,R) + \mathbb{P}(B,B) + \mathbb{P}(G,G) = \frac{22}{56}$$

Examples- Training

Example6-Solution: - Using counting: This operation is an arrangement without repetition $\Rightarrow$ The number of all possible cases is $\mathbb{A}_2^8 = \frac{8!}{(8-2)!} = \frac{8!}{6!} = 56 \rightarrow |\Omega| = 56$

- Event A:{Taking 2 red balls from 5 balls successively}

$$|A| = \mathbb{A}_2^5 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = 20 \Rightarrow \mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{20}{56}$$

- Event B:{Taking 2 blue balls from 2 balls successively}

$$|B| = \mathbb{A}_2^2 = \frac{2!}{(2-2)!} = \frac{2!}{0!} = 2 \Rightarrow \mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{2}{56}$$

- Event C:{Taking 2 green balls from 1 balls successively} $\rightarrow |C| = 0$

$$\Rightarrow \mathbb{P}(C) = \frac{|C|}{|\Omega|} = \frac{0}{56} = 0 \rightarrow \text{Impossible event.}$$

- The probability of getting two balls of the same color $\rightarrow$ {(R,R) Or (B,B) Or (G,G)}

$$\Rightarrow \mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) = \frac{20}{56} + \frac{2}{56} + \frac{0}{56} = \frac{22}{56}.$$

Examples- Training

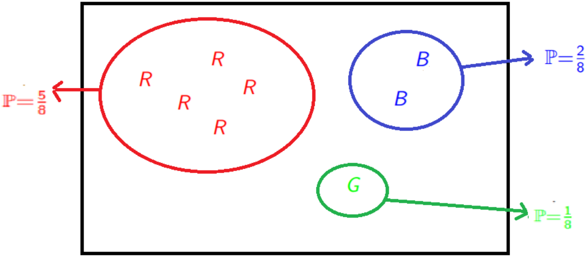
Example6-bis:

In an urn, there are 5 red balls (R), 2 blue balls (B) and 1 green ball (G), indistinguishable by touch. Two balls are drawn simultaneously (at the same time). We want to determine the probability of drawing two balls of the same color.

1. What is the probability of having: the couple (R, R), the couple (B, B), the couple (G, G).
2. Deduce the probability of drawing two balls of the same color.

Examples- Training

Example6-bis-Solution: $\underbrace{R\ R\ R\ R\ R}_{\mathbb{P}(R)=\frac{5}{8}}$ $\underbrace{B\ B}_{\mathbb{P}(B)=\frac{2}{8}}$ $\underbrace{G}_{\mathbb{P}(G)=\frac{1}{8}}$



- This operation is a combination without repetition $\Rightarrow$ The number of all possible cases is $C_2^8 = \frac{8!}{2!(8-2)!} = \frac{8!}{2!6!} = 28 \rightarrow |\Omega| = 28$

Examples- Training

Example6-bis-Solution:

- Event A:{Taking 2 red balls from 5 balls, simultaneously}

$$|A| = C_2^5 = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = 10 \Rightarrow \mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{10}{28}$$

- Event B:{Taking 2 blue balls from 2 balls simultaneously}

$$|B| = C_2^2 = \frac{2!}{2!(2-2)!} = \frac{2!}{2!0!} = 1 \Rightarrow \mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{1}{28}$$

- Event C:{Taking 2 green balls from 1 balls, simultaneously} $\rightarrow\ |C| = 0$

$$\Rightarrow \mathbb{P}(C) = \frac{|C|}{|\Omega|} = \frac{0}{56} = 0 \rightarrow \text{impossible event}$$

- The probability of getting two balls of the same color $\rightarrow$ {(R,R) Or (B,B) Or (G,G)}

$$\Rightarrow \mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) = \frac{10}{28} + \frac{1}{28} + \frac{0}{28} = \frac{11}{28}$$

Examples- Training

Example7:

A student gets dressed very quickly in the morning and takes, at random from the pile of clothes: pant, a T-shirt, a pair of socks. That day there are

- 6 pants including 4 black
- 7 T-shirts including 4 black
- 10 pairs in the wardrobe of socks, including 5 black pairs.

- How many ways are there to dress?

- What are the probabilities of the following events:

A: It is all in black;

B: Only one piece is black out of all three.

Examples- Training

Example7-Solution:

- A typical outcome is (Pant, T-shirt, pair of Socks) = (P, T, S)

$\rightarrow$ There is no repetition (I cannot wear two pants at the same time)

$\rightarrow$ The order doesn't matter

$\rightarrow$ The number of all possible outfits is

$$C_1^6 \ C_1^7 \ C_1^{10} = \frac{6!}{1!5!} \times \frac{7!}{1!6!} \times \frac{10!}{1!9!} = 6 \times 7 \times 10 = 420$$

- Event A: It is all in black (We should combine only the black items with each other)

$$|A| = C_1^4 \ C_1^4 \ C_1^5 = 4 \times 4 \times 5 = 80 \ \Rightarrow \ \mathbb{P}(A) = \frac{|A|}{420} = \frac{80}{420} = 0.1905.$$

- Event B: Only one piece is black out of all three.

N_1 = The pant is in black, N_2 = The T-shirt is in black, N_3 = The socks are in black

$$\begin{aligned} B &= (N_1 \cap N_2^c \cap N_3^c) \cup (N_1^c \cap N_2 \cap N_3^c) \cup (N_1^c \cap N_2^c \cap N_3) \\ |B| &= (C_1^4 \times C_1^3 \times C_1^5) + (C_1^2 \times C_1^4 \times C_1^5) + (C_1^2 \times C_1^3 \times C_1^5) \\ &= (4 \times 3 \times 5) + (2 \times 4 \times 5) + (2 \times 3 \times 5) = 130 \\ \Rightarrow \quad & \mathbb{P}(B) = \tfrac{|B|}{420} = \tfrac{130}{420} = 0.3095. \end{aligned}$$

Examples- Training

Example8:

You want to order a pizza. If you have a choice of 7 different toppings, how many different pizzas can be ordered?

Solution:

$$
\begin{array}{cccccccccl}
C_0^7+ & C_1^7+ & C_2^7+ & C_3^7+ & C_4^7+ & C_5^7+ & C_6^7+ & C_7^7 & & = 1+7+21+35+35+21+7+1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & = 128 = 2^7 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & & \leftarrow \text{number of toppings}
\end{array}
$$

Examples- Training

Example9: A student should solve a quiz of 10 questions where the answers are true (T) or false (F). Suppose that the student will randomly answer to these questions

1) How many possible answer he can make.
   - The number of all the possible answers is $|\Omega| = 2^{10} = 1024$.
2) In how many ways he will answer 6 true and 4 false.
   - A= {There are 6 (T) answers and 4 (F) answers} $\rightarrow |A| = C_6^{10} = \dfrac{10!}{6!4!} = 210$.
3) Suppose that the probability that this student will answer (T) is $p = 0.8$ (i.e; he will answer (F) with probability 0.2). What is the probability that he will answer 6 times (T)?
   - $\mathbb{P}(A) = C_6^{10} p^6 (1-p)^4 = 210.(0.8)^6.(0.2)^4 = 0.0881$
4) If the probability that he will answer (T) is $p = 0.5$ (i.e; he will answer (F) with probability 0.5). What is the probability that he will answer 6 times (T)?
   - $\mathbb{P}(A) = C_6^{10} p^6 (1-p)^4 = 210.(0.5)^6.(0.5)^4 = 0.2051 = \dfrac{|A|}{|\Omega|} = \dfrac{210}{1024} = 0.2051. \rightarrow$

   Since (T) and (F) are equally likely.

Examples- Training

5) If there are 6 (T) answers in the quiz. What is the probability to get 100% as a score in the both above cases.
$\rightarrow \mathbb{P}(\text{Getting } 100\ \%) = 1 \times p^6(1-p)^4 = 1.(0.8)^6.(0.2)^4 = (4.19).10^{-4}$
$\rightarrow \mathbb{P}(\text{Getting } 100\ \%) = 1 \times p^6(1-p)^4 = 1.(0.5)^6.(0.5)^4 = (9.76).10^{-4} = \dfrac{1}{1024}$

6) Assuming that the student knows in advance that there are 6 answers (T), what is the probability that he will get 100%?

$$\mathbb{P}(\text{Getting } 100\ \% | A) = \frac{\mathbb{P}(\text{Getting } 100\ \% \cap A)}{\mathbb{P}(A)} = \frac{1 \times p^6(1-p)^4}{210 \times p^6(1-p)^4} = \frac{1}{210} = 0.0048$$

$\rightarrow$ The probability increased from $(4.19).10^{-4}$ to 0.0048

Examples- Training

Example 10: (How many bit strings:)

A 6-bit string is sent over a network. The valid set of strings recognized by the receiver must either start with "01" or end with "10". How many such strings are there?

Solution:

- Let A={Strings start with "01" } $\rightarrow |A| = 2^4 = 16$
- Let B={Strings end with "10"} $\rightarrow |B| = 2^4 = 16$

$\rightarrow |A \cap B| = 2^2 = 4$

2^4 start with 01	2^4 end with 10
010000	000010
010001	000110
010010	001010
010011	001110
010100	010010
010101	010110
010110	011010
010111	011110
011000	100010
011001	100110
011010	101010
011011	101110
011100	110010
011101	110110
011110	111010
011111	111110
Set A	Set B

$|A \cup B| = |A| + |B| - |A \cap B| = 16 + 16 - 4 = 28$

Examples- Training

Example 11: (unique 6-digit passcodes with six smudges:)

• How many unique 6-digit passcodes are possible if a phone password uses each of six distinct numbers shown in the figure.

- It is a permutation $\rightarrow 6! = 720$ passcodes.

• How many unique 6-digit passcodes are possible if a phone password is some ordered subset of any six distinct digits.

- It is an arrangement $\rightarrow \mathbb{A}_6^{10} = \dfrac{10!}{(10-6)!} = \dfrac{10!}{4!} = 10 \times 9 \times 8 \times 7 \times 6 \times 5 = 151200$ passcodes.

Examples- Training

Example 12:

How many strings containing two times 1 and three times 0 are there?

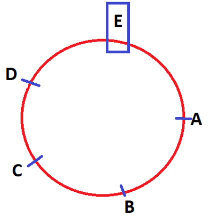
$\rightarrow \{10100, 01001, \ldots\} \rightarrow$ It is a permutation with repetition: $\dfrac{5!}{2!3!} = 10$

Example 13:

How many ways can 5 people sit around a table?

Solution:

- We fix one element as a reference and permutes the others $\rightarrow$ 4!



The rule is: $\boxed{\#\text{ circular permutation of n elements is } (n-1)!}$

Examples- Training

Example13:

What is the probability p_n that in a group of n persons, chosen at random, at least two people have the same birthday (we will assume that the year always has 365 days, all equally likely).

- Show that for $n > 23$, we have $p_n > \frac{1}{2}$

Examples- Training

Example11-Solution:

1) If $n > 366$: We clearly have $p_n = 1$

$\rightarrow$ (if 366 people are to be associated with 365 anniversary dates, then at least 2 people are to be associated with the same anniversary date)

2) For $2 \leq n \leq 365$:

There are $(365)^n$ possible distributions of birthdays (possible cases) $\rightarrow$ it is an arrangement with repetition $\Rightarrow$ Among these distributions, there are

$$365 \times 364 \times 363.... \times (365 - n + 1) = \frac{365!}{(365-n)!} = \mathbb{A}_n^{365}$$ distributions such that the

anniversary dates are two by two distinct.

- Let the event A: $\{$The anniversary dates are two by two distinct$\}$

$$\Rightarrow \mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{365 \times 364 \times 363.... \times (365 - n + 1)}{(365)^n} = q_n \Rightarrow p_n = 1 - q_n$$

Examples- Training

Example11-Solution:

$$p_n = 1 - \frac{365 \times 364 \times 363.... \times (365 - n + 1)}{(365)^n} = 1 - \prod_{k=1}^{n-1} \frac{365 - k}{365} = 1 - \prod_{k=1}^{n-1} \left[1 - \frac{k}{365}\right]$$

We have

$$\ln\left(\prod_{k=1}^{n-1}\left[1 - \frac{k}{365}\right]\right) = \sum_{k=1}^{n-1} \ln\left(1 - \frac{k}{365}\right) \Rightarrow -\ln\left(\prod_{k=1}^{n-1}\left[1 - \frac{k}{365}\right]\right) = \sum_{k=1}^{n-1} -\ln\left(1 - \frac{k}{365}\right)$$

We know that for $0 < x < 1$, so $\ln(1 - x) \leq -x \Rightarrow \ln\left(1 - \frac{k}{365}\right) \leq -\frac{k}{365}$

$$\Rightarrow \sum_{k=1}^{n-1} -\ln\left(1 - \frac{k}{365}\right) \geq \sum_{k=1}^{n-1} \frac{k}{365} = \frac{1}{365} \sum_{k=1}^{n-1} k = \frac{1}{365} \times \frac{n(n-1)}{2} = \frac{n(n-1)}{730}$$

Examples- Training

- We want to describe n such that

$$p_n = 1 - \prod_{k=1}^{n-1}\left[1 - \frac{k}{365}\right] \geq \frac{1}{2} \quad \ldots \quad (1)$$

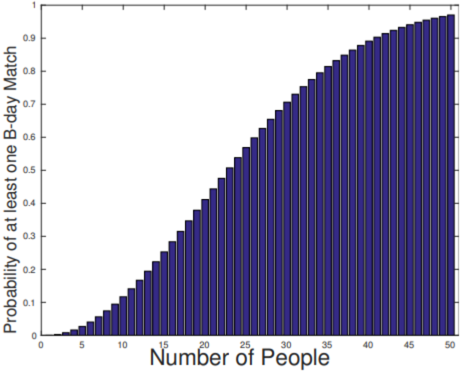
$$1-\prod_{k=1}^{n-1}\left[1 - \frac{k}{365}\right] \geq \frac{1}{2} \Rightarrow -\prod_{k=1}^{n-1}\left[1 - \frac{k}{365}\right] \geq \frac{-1}{2} \Rightarrow -\ln\left(\prod_{k=1}^{n-1}\left[1 - \frac{k}{365}\right]\right) \geq -\ln\left(\frac{1}{2}\right) = \ln(2)$$

The inequality (1) is satisfied if

$$\frac{n(n-1)}{730} \geq \ln(2) \;\Rightarrow\; n^2 - n \geq 730\ \ln(2) \Rightarrow n \geq \frac{1 - \sqrt{1 + 2920\ln(2)}}{2} = 22.99 \Rightarrow n \geq 23$$

Examples- Training

$$p_n = 1 - \frac{365 \times 364 \times 363.... \times (365 - n + 1)}{(365)^n}$$



- In this class $p_{130} \approx 1$