Probability and Random Variables (ECE313/ECE317)

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Fall 2023

Probability: Sample set

Probability is a law which is assigned to experiences where the result is uncertain.

- The list of all the possible outcomes is a set denoted by Ω and called the "Sample set".

Example

- Experiment: Flipping a coin with two faces: Head(H) and Tail (T)
- The set of possible outcomes $\Omega = \{H, T\}$.
- Experiment: Flipping a coin with two faces three times

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Probability: Sample set

Example

- Experiment: Tossing a dice with 6 faces
- The set of possible outcomes $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- Experiment: Tossing a dice with 6 faces, two times
- The set of possible outcomes

$$\Omega = \{(1,1), (1,2), \ldots, (6,6)\} = 36 \text{ elements (see figure)}$$

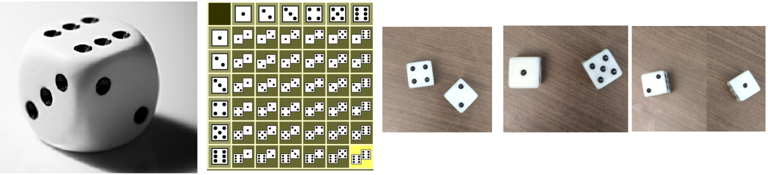


Figure: Tossing a dice

Probability: Axioms

Which outcomes are more likely to occur and which ones are less likely to occur ?

$\Rightarrow$ We do that by assigning probability ($\mathbb{P}$) to the different outcomes.

- Event: A subset of the sample space $\Rightarrow$ Probability is assigned to events.

Set Algebra	**Probability**
Set	Event
Universal set	Sample space
Element	Outcome

Figure: The terminology of set theory and probability

- The list of outcomes must be:

- Collectively exhaustive: The union of all the outcomes is the total sample set.

Probability: Axioms

- Axioms: (basic properties of the probability)

$$\left.\begin{array}{l} -\text{Nonnegativity: } \mathbb{P}(A) \geq 0 \\ \\ -\text{Normalization: } \mathbb{P}(\Omega) = 1 \end{array}\right\} \Rightarrow \quad 0 \leq \mathbb{P}(A) \leq 1$$

- The likelihood of any event to occur is a number between 0 and 1,
  - 0 indicates the impossibility of the event.
  - 1 indicates the certainty of the even
- **Union of events:** $A \cup B$ means that "A occurs **OR** B occurs".
- **Intersection of events:** $A \cap B$ means that "A occurs **AND** B occurs".

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

- **Additivity:** If $A \cap B = \emptyset$ (A and B are disjoint events), then

$$\mathbb{P}(\mathrm{A} \cup \mathrm{B}) = \mathbb{P}(\mathrm{A}) + \mathbb{P}(\mathrm{B}).$$

Probability: Axioms

- Consequences:

1) $\Omega = \Omega \cup \emptyset$, and $\Omega \cap \emptyset = \emptyset$, as a consequence:

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(\Omega \cup \emptyset) = \mathbb{P}(\Omega) + \mathbb{P}(\emptyset) = 1 + \mathbb{P}(\emptyset) \Rightarrow \mathbb{P}(\emptyset) = 0$$

2) $A \cup A^c = \Omega$, and $A \cap A^c = \emptyset$, as a consequence

- $\mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c) = 1 \Rightarrow \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

3) If $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

$$B = (B \cap A^c) \cup A \Rightarrow \mathbb{P}(B) = \mathbb{P}((B \cap A^c) \cup A) = \mathbb{P}(B \cap A^c) + \mathbb{P}(A) \geq \mathbb{P}(A).$$

Probability: Axioms
- Consequences:

4) $\mathbb{P}(A \cup B \cup C) = \mathbb{P}((A \cup B) \cup C) = \mathbb{P}(A \cup B) + \mathbb{P}(C) - \mathbb{P}((A \cup B) \cap C)$

$$= \underbrace{\mathbb{P}(A \cup B)} + \mathbb{P}(C) - \underbrace{\mathbb{P}[(A \cap C) \cup (B \cap C)]}$$

$$= \underbrace{\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)} + \mathbb{P}(C) - \underbrace{[\mathbb{P}(A \cap C) + \mathbb{P}(B \cap C) - \mathbb{P}((A \cap C) \cap (B \cap C))]}$$

▷ If A, B and C are mutually exclusive, then

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(\underbrace{A \cap B}_{\emptyset}) + \mathbb{P}(C) - \mathbb{P}(\underbrace{A \cap C}_{\emptyset}) - \mathbb{P}(\underbrace{B \cap C}_{\emptyset}) + \mathbb{P}(\underbrace{A \cap B \cap C}_{\emptyset})$$

$$= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C)$$

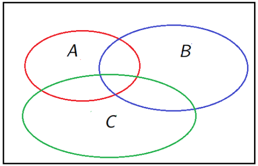
▷ If $A_1, A_2, \ldots, A_k$ are mutually exclusive, then

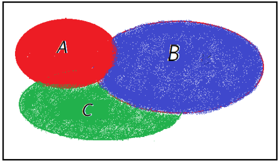
$$\mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_k) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \ldots + \mathbb{P}(A_k).$$

Probability: Axioms

5) $A \cup B \cup C = A \cup (B \cap A^c) \cup (C \cap A^c \cap B^c)$

$\rightarrow$ is a union of **disjoints** sets: $A \cap (B \cap A^c) \cap (C \cap (A^c \cap B^c)) = \emptyset$





$$A \cup B \cup C = A \cup \; (B \cap A^c) \cup (C \cap (A^c \cap B^c)) \leftarrow$$ **union of disjoint sets**

red part   blue part   green part

$$\Rightarrow \; \mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B \cap A^c) + \mathbb{P}(C \cap A^c \cap B^c)$$

6) $(A^c)^c = A \; \Rightarrow \; \mathbb{P}((A^c)^c) = \mathbb{P}(A)$

Probability: Axioms

Example

- Let $\mathbb{P}(A) = 0.6, \quad \mathbb{P}(B^c \cup C) = 0.5, \quad \mathbb{P}(A \cap B^c) = 0.4, \quad A \cap C = \emptyset$

$$\begin{aligned}
\mathbb{P}[A \cup (B^c \cup C)] &= \mathbb{P}(A) + \mathbb{P}(B^c \cup C) - \mathbb{P}(A \cap (B^c \cup C)) \\
&= \mathbb{P}(A) + \mathbb{P}(B^c \cup C) - \mathbb{P}((A \cap B^c) \cup (A \cap C)) \\
&= \mathbb{P}(A) + \mathbb{P}(B^c \cup C) - \mathbb{P}((A \cap B^c) \cup \emptyset) \\
&= \mathbb{P}(A) + \mathbb{P}(B^c \cup C) - \mathbb{P}(A \cap B^c) \\
&= 0.6 + 0.5 - 0.4 = 0.7
\end{aligned}$$

Probability: Axioms

Example

- Let $\mathbb{P}(A) = 0.5, \quad \mathbb{P}(A^c \cap B) = 0.3$ (i.e; $\mathbb{P}(B - A) = 0.3$)

$$\begin{aligned}
\mathbb{P}(A \cup B) = \mathbb{P}[A \cup (A^c \cap B)] &= \mathbb{P}(A) + \mathbb{P}(A^c \cap B) - \mathbb{P}(A \cap A^c \cap B) \\
&= \mathbb{P}(A) + \mathbb{P}(A^c \cap B) - \mathbb{P}(\emptyset \cap B) \\
&= \mathbb{P}(A) + \mathbb{P}(A^c \cap B) - \mathbb{P}(\emptyset) \\
&= \mathbb{P}(A) + \mathbb{P}(A^c \cap B) + 0 \\
&= 0.5 + 0.3 + 0 = 0.8
\end{aligned}$$

Probability: Axioms

Example

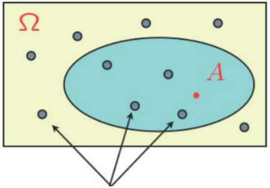
- Let $\mathbb{P}(A) = 0.4, \ \ C \subset A$

$$\begin{aligned} \mathbb{P}[A \cup (B^c \cap C)] &= \mathbb{P}(A) + \mathbb{P}(B^c \cap C) - \mathbb{P}(A \cap B^c \cap C) \\ &= \mathbb{P}(A) + \mathbb{P}(B^c \cap C) - \mathbb{P}(B^c \cap C) \\ &= \mathbb{P}(A) = 0.4 \end{aligned}$$

Probability: Discrete uniform law:

- Assume Ω consist of n equally likely elements (n = cardinal of Ω, and we denote $|\Omega| = n$).
- Assume the event A consists of k elements (k = cardinal of A, and we denote $|A| = k$).

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{k}{n} = k \times \frac{1}{n} \Rightarrow \frac{1}{n} = \text{probability of each element}$$



$$\mathbb{P} = \frac{1}{n}$$

Figure: Discrete sample set

Probability: Examples

Example

- Flipping a coin $\Rightarrow \; \Omega = \{H, T\}$
- Event: $A = \{H\}$

$$\mathbb{P}(A) = \mathbb{P}(H) = \frac{1}{2}$$

- Event: $A = \{T\}$

$$\mathbb{P}(A) = \mathbb{P}(T) = \frac{1}{2}$$

- Event: $A = \{H, T\}$

$$\mathbb{P}(A) = \mathbb{P}(\{H, T\}) = \mathbb{P}(\Omega) = 1$$

Probability: Examples

Example

- Tossing a die $\Rightarrow \;\; \Omega = \{1,2,3,4,5,6\} = 6$ elements
- Event: $A = \{1\}$

$$\mathbb{P}(A) = \mathbb{P}(1) = \frac{1}{6} = 0.166$$

- Event: $A = \{3\}$

$$\mathbb{P}(A) = \mathbb{P}(3) = \frac{1}{6} = 0.166$$

- Event: $A = \{1,3,4,6\}$

$$\mathbb{P}(\{1,3,4,6\}) = \mathbb{P}(\{1\} \cup \{3\} \cup \{4\} \cup \{6\}) = \mathbb{P}(\{1\}) + \mathbb{P}(\{3\}) + \mathbb{P}(\{4\}) + \mathbb{P}(\{6\})$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3} = 0.666$$

Probability: Examples

Example

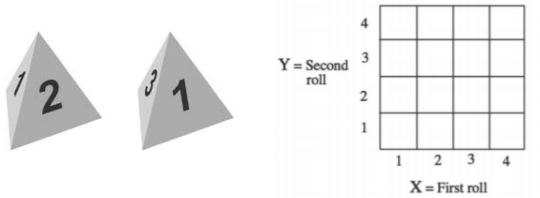


Figure: Two rolls of a tetrahedral die

$$\Omega = \{\text{All } (X, Y)\} = \{(1,1), (1,2), \ldots (4,4)\} \Rightarrow \text{16 elements}$$

$$\mathbb{P}(\text{each } (X, Y)) = \frac{1}{16}$$

Probability: Examples

- Event: $A = \{(X, Y) : X = 1\}$, $A = \{(1,1), (1,2), (1,3), (1,4)\}$

$$\mathbb{P}(A) = \mathbb{P}((1,1), (1,2), (1,3), (1,4)) = 4 \times \frac{1}{16} = \frac{4}{16} = \frac{1}{4}$$

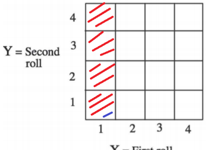
- Event: $Z = \{(X, Y) : \min(X, Y) = 4\}$, $Z = \{(4,4)\}$

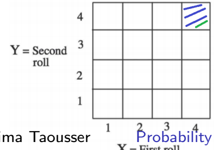
$$\mathbb{P}(Z) = \mathbb{P}((4,4)) = \frac{1}{16}.$$

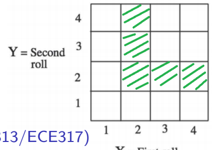
- Event:

$Z = \{(X, Y) : \min(X, Y) = 2\}$, $Z = \{(2,2), (2,3), (2,4), (3,2), (4,2)\}$

$$\mathbb{P}(Z) = \mathbb{P}((2,2), (2,3), (2,4), (3,2), (4,2)) = 5 \times \frac{1}{16} = \frac{5}{16}.$$







Probability: Continuous law:

- Consider the continuous sample set Ω.
- Let the event A which is a subset of Ω.

$$\mathbb{P}(A) = \frac{\text{area(A)}}{area(\Omega)}.$$

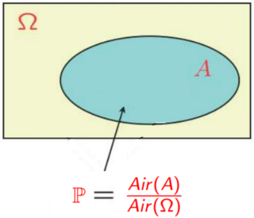


Figure: Continuous sample set

▷ Uniform probability law: Probability is computed according to the area since the number of elements of the set is not countable

Probability: Examples

Example

$$\Omega = \{(x, y) : 0 \le x \le 1, \quad 0 \le y \le 1\}$$

$$area(\Omega) = 1 \times 1 = 1$$

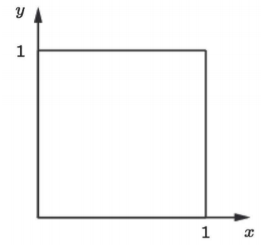
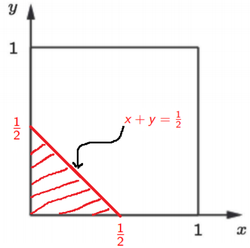


Figure: Continuous sample set (the unit square)

Probability: Examples

- Event: $A = \{(x, y) : x + y \leq \frac{1}{2}\}$



$$\mathbb{P}(A) = \mathbb{P}(\{(x, y) : x + y \leq \frac{1}{2}\}) = \frac{area(A)}{area(\Omega)} = \frac{\frac{(\frac{1}{2} \times \frac{1}{2})}{2}}{1} = \frac{1}{8}$$

- Event: $A = \{x = 0.5, \quad y = 0.3\}$

$$\Rightarrow \mathbb{P}(A) = \mathbb{P}(\{(0.5, 0.3)\}) = \frac{area(A)}{area(\Omega)} = \frac{0}{1} = 0$$

Probability computation steps

- Specify the sample space: (come up with a list of all possible outcomes).
- Specify a probability law:
  (by assigning probabilities to subsets of the sample set according to our believe to be likely and to be unlikely).
- Identify an event of interest.
- Calculate.

Probability: Examples

Example1: - Experiment: Flipping a coin with two faces (H, T) three times

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \rightarrow |\Omega| = 8.$$

- Event A: We get only one Head

$$A = \{HTT, THT, TTH\} \rightarrow |A| = 3 \rightarrow \mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{3}{8}$$

- Event B: We get at least two Tails

$$B = \{HTT, THT, TTH, TTT\} \rightarrow |B| = 4 \rightarrow \mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{4}{8}$$

- Event C: We get at most two Heads

$$C = \{HHT, HTH, HTT, THH, THT, TTH, TTT\} \rightarrow |C| = 7 \rightarrow \mathbb{P}(C) = \frac{|C|}{|\Omega|} = \frac{7}{8}$$

Probability: Examples

1) What is the probability that at least one of the events A, B and C occurs?
(A occurs **OR** B occurs **OR** C occurs) $\rightarrow$ $(A \cup B \cup C)$

$$A \cup B \cup C = \{HTT, THT, TTH, TTT, HHT, HTH, THH\}$$

$$\rightarrow \ |A \cup B \cup C| = 7 \ \rightarrow \mathbb{P}(A \cup B \cup C) = \frac{7}{8}$$

2) What is the probability that none of the events A, B and C occurs?
(not A **And** not B **And** not C) $\rightarrow$ $(A^c \cap B^c \cap C^c) = (A \cup B \cup C)^c$

$$(A \cup B \cup C)^c = \{HHH\} \ \rightarrow \ |(A \cup B \cup C)^c| = 1 \ \rightarrow \mathbb{P}((A \cup B \cup C)^c) = \frac{1}{8}$$

Or

$$\mathbb{P}((A \cup B \cup C)^c) = 1 - \mathbb{P}(A \cup B \cup C) = 1 - \frac{7}{8} = \frac{1}{8}$$

Probability: Examples

3) What is the probability that all the three events A, B and C occur?
(A occurs **And** B occurs **And** C occurs) $\rightarrow$ $(A \cap B \cap C)$

$$(A\cap B\cap C) = \{HTT, THT, TTH\} \rightarrow |(A\cap B\cap C)| = 3 \rightarrow \mathbb{P}((A\cap B\cap C)) = \frac{3}{8}$$

4) What is the probability that exactly one of the events A, B, C occurs?
(A occurs **And** not B **And** not C) **Or** (B occurs **And** not A **And** not C) **Or** (C occurs **And** not A **And** not B)

$$[A\cap B^c \cap C^c] \cup [B \cap A^c \cap C^c] \cup [C \cap A^c \cap B^c]$$

$$[A\cap B^c \cap C^c] = \emptyset, \;\; [B \cap A^c \cap C^c] = \emptyset, \;\; [C \cap A^c \cap B^c] = \{HHT, HTH, THH\}$$

$$\rightarrow \; |[A\cap B^c \cap C^c] \cup [B \cap A^c \cap C^c] \cup [C \cap A^c \cap B^c]| = 3 \; \Rightarrow \; \mathbb{P} = \frac{3}{8}$$

Probability: Examples

5) What is the probability that the events A, B occur but not C?
(A occurs **And** B occurs **And** not C)

$$A \cap B \cap C^c$$

$A \cap B \cap C^c = \emptyset \rightarrow |A \cap B \cap C^c| = 0 \rightarrow \mathbb{P}(A \cap B \cap C^c) = 0 \rightarrow$ impossible event

Probability: Examples

6) What is the probability that at most one of the events A, B , C occurs?
(A occurs **And** not B **And** not C) **Or** (B occurs **And** not A **And** not C) **Or** (C occurs **And** not A **And** not B, **Or** none of them)

$$[A \cap B^c \cap C^c] \cup [B \cap A^c \cap C^c] \cup [C \cap A^c \cap B^c] \cup [A^c \cap B^c \cap C^c]$$

$$[A \cap B^c \cap C^c] = \emptyset, \quad [B \cap A^c \cap C^c] = \emptyset, \quad [C \cap A^c \cap B^c] = \{HHT, HTH, THH\}$$

$$[A^c \cap B^c \cap C^c] = \{HHH\}$$

$$[A \cap B^c \cap C^c] \cup [B \cap A^c \cap C^c] \cup [C \cap A^c \cap B^c] \cup [A^c \cap B^c \cap C^c] = \{HHT, HTH, THH, HHH\}$$

$$\rightarrow \ |[A \cap B^c \cap C^c] \cup [B \cap A^c \cap C^c] \cup [C \cap A^c \cap B^c] \cup [A^c \cap B^c \cap C^c]| = 4 \ \Rightarrow \ \mathbb{P} = \frac{4}{8} = \frac{1}{2}$$

Probability: Examples

Example2:

- Alice and Bob each choose at random a number in the interval $[0, 2]$.

Consider the following events

A) Both numbers are greater than $\frac{1}{3}$

B) At least one of the numbers is greater than $\frac{1}{3}$

C) The two numbers are equal

D) Alice's number is greater than $\frac{1}{3}$

E) The magnitude of the difference of the two number is greater than $\frac{1}{3}$

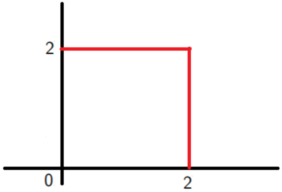
- Find the probabilities:

$\mathbb{P}(A)$, $\mathbb{P}(B)$, $\mathbb{P}(C)$, $\mathbb{P}(D)$, $\mathbb{P}(E)$, $\mathbb{P}(A \cap D)$, $\mathbb{P}(D \cap E)$

Probability: Examples

$$\Omega = \{(x, y) : 0 \leq x \leq 2, \quad 0 \leq y \leq 2\}$$

- Uniform probability law:

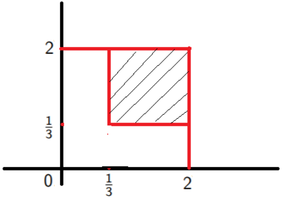


$$area(\Omega) = 2 \times 2 = 4$$

Probability: Examples

A) Both numbers are greater than $\frac{1}{3}$

$$A = \{(x, y) : x \geq \frac{1}{3}, \text{ and } y \geq \frac{1}{3}\}$$



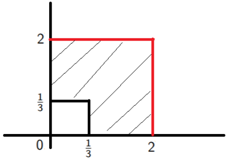
$$area(A) = (2 - \frac{1}{3}) \times (2 - \frac{1}{3}) = \frac{5}{3} \times \frac{5}{3} = \frac{25}{9}$$

$$\mathbb{P}(A) = \frac{area(A)}{area(\Omega)} = \frac{\frac{25}{9}}{4} = \frac{25}{36}$$

Probability: Examples

B) At least one of the numbers is greater than $\frac{1}{3}$

$$B = \{(x, y) : x \geq \frac{1}{3}, \text{ or } y \geq \frac{1}{3}\}$$



$$B^c = \{(x, y) : x \leq \tfrac{1}{3}, \text{ and } y \leq \tfrac{1}{3}\} \quad \Rightarrow \mathbb{P}(B^c) = \frac{area(B^c)}{area(\Omega)} = \frac{\frac{1}{3} \times \frac{1}{3}}{4} = \frac{\frac{1}{9}}{4} = \frac{1}{36}$$

$$\Rightarrow \mathbb{P}(B) = 1 - \mathbb{P}(B^c) = 1 - \frac{1}{36} = \frac{35}{36}$$

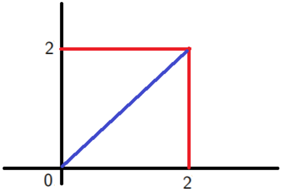
Or

$$area(B) = 4 - area(B^c) = 4 - \frac{1}{9} = \frac{35}{9} \Rightarrow \mathbb{P}(B) = \frac{area(B)}{area(\Omega)} = \frac{\frac{35}{9}}{4} = \frac{35}{36}$$

Probability: Examples

C) The two numbers are equal

$$C = \{(x, y) : x = y\}$$



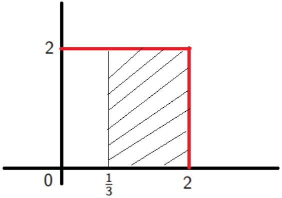
$$area(C) = 0 \;\rightarrow\; \text{the area of a line} \;=\; 0$$

$$\mathbb{P}(C) = \frac{area(C)}{area(\Omega)} = \frac{0}{4} = 0$$

Probability: Examples

D) Alice's number is greater than $\frac{1}{3}$

$$D = \{(x, y) : x \geq \frac{1}{3}\}$$

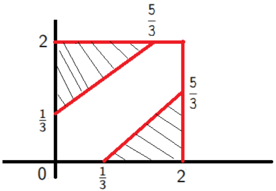


$$area(D) = (2 - \frac{1}{3}) \times 2 = \frac{10}{3} \Rightarrow \mathbb{P}(D) = \frac{area(D)}{area(\Omega)} = \frac{\frac{10}{3}}{4} = \frac{10}{12} = \frac{5}{6}$$

Probability: Examples

E) The magnitude of the difference of the two numbers is greater than $\frac{1}{3}$

$$E = \{(x,y) : |x-y| > \frac{1}{3}\} = (x,y) : \begin{cases} x - y > \frac{1}{3} \\ \text{OR} \\ x - y < \frac{-1}{3} \end{cases} \Rightarrow \begin{cases} y < x - \frac{1}{3} \\ \text{OR} \\ y > x + \frac{1}{3} \end{cases}$$



$$area(E) = \frac{\frac{5}{3} \times \frac{5}{3}}{2} + \frac{\frac{5}{3} \times \frac{5}{3}}{2} = \frac{25}{9} \Rightarrow \mathbb{P}(E) = \frac{area(E)}{area(\Omega)} = \frac{\frac{25}{9}}{4} = \frac{25}{36}$$

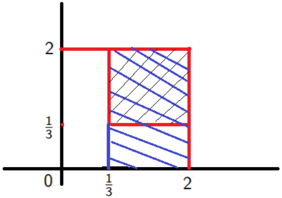
- Note that, $E^c = \{(x,y) : |x-y| \leq \frac{1}{3}\} = \{(x,y) : x - y \leq \frac{1}{3} \text{ and } x - y \geq \frac{-1}{3}\}$

$$\Rightarrow \mathbb{E} = 1 - \mathbb{P}(E) = 1 - \frac{25}{36} = \frac{11}{36}$$

Probability: Examples

- Both numbers are greater than $\frac{1}{3}$ **and** Alice's number is greater than $\frac{1}{3}$

$$A \cap D = \{(x,y) : x \geq \frac{1}{3} \textbf{ and } x \geq \frac{1}{3} \textbf{ and } y \geq \frac{1}{3}\} = \{(x,y) : x \geq \frac{1}{3} \textbf{ and } y \geq \frac{1}{3}\} = A$$

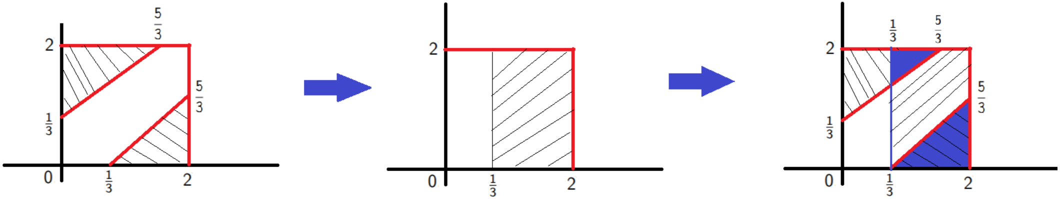


$$\mathbb{P}(A \cap D) = \mathbb{P}(A) = \frac{25}{36}$$

Probability: Examples

- Alice's number is greater than $\frac{1}{3}$ and the magnitude of the difference of the two numbers is greater than $\frac{1}{3}$

$$D \cap E = \{(x, y) : x \geq \frac{1}{3} \text{ and } x - y > \frac{1}{3} \text{ and } x - y < \frac{-1}{3}\}$$



$$area(D \cap E) = \frac{(\frac{5}{3} - \frac{1}{3}) \times (\frac{5}{3} - \frac{1}{3})}{2} + \frac{\frac{5}{3} \times \frac{5}{3}}{2} = \frac{\frac{4}{3} \times \frac{4}{3}}{2} + \frac{\frac{5}{3} \times \frac{5}{3}}{2} = \frac{41}{18}$$

$$\mathbb{P}(D \cap E) = \frac{area(D \cap E)}{area(\Omega)} = \frac{\frac{41}{18}}{4} = \frac{41}{72}$$

Examples- Training

Problem 1: Given the two events A and B such that:

$$\mathbb{P}(A) = 0.3, \quad \mathbb{P}(B) = 0.6, \quad \mathbb{P}(A \cap B) = 0.18$$

- Find

i) $\mathbb{P}(A \text{ or } B)$

ii) $\mathbb{P}(A \text{ and not} B)$

iii) $\mathbb{P}(\text{neither } A \text{ nor } B)$

Problem 1-Solution:

i) $\mathbb{P}(A \text{ or } B) = \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.3 + 0.6 - 0.18 = 0.72$

ii) $\mathbb{P}(A \text{ and not} B) = \mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = 0.3 - 0.18 = 0.12$

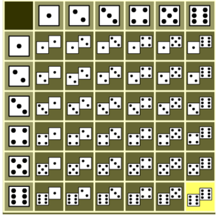
- Method 2: $A \cup B = B \cup (A \cap B^c) \Rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(B) + \mathbb{P}(A \cap B^c) \Rightarrow \mathbb{P}(A \cap B^c) = \mathbb{P}(A \cup B) - \mathbb{P}(B) = 0.72 - 0.6 = 0.12$

iii) $\mathbb{P}(\text{neither } A \text{ nor } B) = \mathbb{P}(A^c \cap B^c) = \mathbb{P}((A \cup B)^c) = 1 - \mathbb{P}(A \cup B) = 1 - 0.72 = 0.28$

Examples- Training

Problem 2:

A die is tossed 2 times. Find the probability of getting an odd number at least one time.



$\Omega = \{(1,2),(1,2),\ldots,(6,6)\} \to |\Omega| = 36$

Event A: Getting an odd number at least one time $\to$ (odd, even) or (odd, odd) or (even, odd) $\to \underbrace{\{1,3,5\}}_{\text{odd}}, \underbrace{\{2,4,6\}}_{\text{even}} \Rightarrow |A| = |(\text{odd, even})| + |(\text{odd, odd})| + |(\text{even, odd})| =$

$(3\times 3) + (3\times 3) + (3\times 3) = 27 \Rightarrow \mathbb{P}(A) = \dfrac{27}{36} = \dfrac{3}{4}$

- Method 2: $|A^c| = |(\text{even, even})| = 3\times 3 = 9 \Rightarrow \mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \dfrac{9}{36} = \dfrac{27}{36}$

Examples- Training

Problem 3: A die is tossed 3 times. Find the probabilities of the following events

A: Getting an even number at least one time.

B: Getting an even number at most one time.

C: Getting an even number exactly one time.

Problem 3-Solution:

$\Omega = \{(1,1,1),(1,1,2),\dots,(6,6,6)\} \rightarrow |\Omega| = 216$

• A: Getting an even number at least one time. $\rightarrow$ (even, odd, even) or (even, even, odd) or (odd, even, odd), $\dots \rightarrow$ The easiest way is to consider the complement:

A^c: Getting no even number $\rightarrow$ (odd, odd, odd) $\rightarrow \underbrace{\{1,3,5\}}_{\text{odd}} \Rightarrow |A^c| = (3\times 3\times 3) = 27$

$$\Rightarrow \mathbb{P}(A^c) = \frac{27}{216} = \frac{1}{8} \Rightarrow \mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \frac{1}{8} = \frac{7}{8} = 0.87$$

• B: Getting an even number at most one time. $\rightarrow$ (even, odd, odd) or (odd, even, odd) or (odd, odd, even) or (odd, odd, odd) $\rightarrow |B| = 4\times(3\times 3\times 3) = 108 \Rightarrow \mathbb{P}(B) = \frac{108}{216} = 0.5$

• C: Getting an even exactly one time. $\rightarrow$ (even, odd, odd) or (odd, even, odd) or (odd, odd, even) $\rightarrow |B| = 3\times(3\times 3\times 3) = 81 \Rightarrow \mathbb{P}(B) = \frac{81}{216} = 0.37$

Examples- Training

Problem 3:

In a survey of 200 people that had just returned from a trip to Europe, the following information was gathered.

- 142 visited England
- 95 visited Italy
- 65 visited Germany
- 70 visited both England and Italy
- 50 visited both England and Germany
- 30 visited both Italy and Germany
- 20 visited all these three countries

a) How many went to England but not Italy or Germany?

b) How many went to exactly one of these three countries?

c) How many went to none of these three countries?

e) Compute the probabilities of the events described in a), b) and c)

Examples- Training

Problem 3-Solution:

$|\Omega| = 200$

- E = people who visited England $\rightarrow n(E) = 142$
- I = people who visited Italy $\rightarrow n(I) = 95$
- G = people who visited Germany $\rightarrow n(G) = 65$
- $E \cap I$ = people who visited both England and Italy $\rightarrow n(E \cap I) = 70$
- $E \cap G$ = people who visited both England and Germany $\rightarrow n(E \cap G) = 50$
- $I \cap G$ = people who visited both Italy and Germany $\rightarrow n(I \cap G) = 30$
- $E \cap I \cap G$ = people who visited all these three countries $\rightarrow n(E \cap I \cap G) = 20$

$$\text{Rule : } \; n(A) = n(A - B) + n(A \cap B) \Rightarrow n(A - B) = n(A) - n(A \cap B)$$

a) How many went to England but not to Italy or Germany? $\rightarrow$
$n(E \cap (I \cup G)^c) = n(E - (I \cup G)) = ?$

$$n(E) = n[E - (I \cup G)] + n[E \cap (I \cup G)] = n(E \cap (I \cup G)^c) + n((E \cap I) \cup (E \cap G))$$

$$\Rightarrow n(E \cap (I \cup G)^c) = n(E) - n((E \cap I) \cup (E \cap G)) = n(E) - [n(E \cap I) + n(E \cap G) - n(E \cap I \cap G)]$$

$$= 142 - [50 + 30 + 20] = 42$$

Examples- Training

Problem 3-Solution:

$|\Omega| = 200$

- E = people who visited England $\to n(E) = 142$
- I = people who visited Italy $\to n(I) = 95$
- G = people who visited Germany $\to n(G) = 65$
- $E \cap I$ = people who visited both England and Italy $\to n(E \cap I) = 70$
- $E \cap G$ = people who visited both England and Germany $\to n(E \cap G) = 50$
- $I \cap G$ = people who visited both Italy and Germany $\to n(I \cap G) = 30$
- $E \cap I \cap G$ = people who visited all these three countries $\to n(E \cap I \cap G) = 20$

b) How many went to exactly one of these three countries?

$\to n[(E \cap I^c \cap G^c) \cup (E^c \cap I \cap G^c) \cup (E^c \cap I^c \cap G)] =$
$n(E \cap I^c \cap G^c) + n(E^c \cap I \cap G^c) + n(E^c \cap I^c \cap G)$

$$= n[E \cap (I \cup G)^c] + n[I \cap (E \cap G)^c] + n[G \cap (E \cup I)^c] = n[E - (I \cup G)] + n[I - (E \cap G)] + n[G - (E \cup I)]$$

$$= [n(E) - n(E \cap (I \cup G))] + [n(I) - n(I \cap (E \cup G))] + [n(G) - n(G \cap (E \cup I))]$$

$$= [n(E) - (n(E \cap I) + n(E \cap G) - n(E \cap I \cap G))] + [n(I) - (n(I \cap E) + n(I \cap G) - n(E \cap I \cap G))]$$

$$+ [n(G) - (n(G \cap I) + n(G \cap E) - n(E \cap I \cap G))] = 42 + [95 - (70 + 30 - 20)] + [65 - (30 + 50 - 20)] = 62$$

Examples- Training

Problem 3-Solution:

$|\Omega| = 200$

- E = people who visited England $\rightarrow n(E) = 142$
- I = people who visited Italy $\rightarrow n(I) = 95$
- G = people who visited Germany $\rightarrow n(G) = 65$
- $E \cap I$ = people who visited both England and Italy $\rightarrow n(E \cap I) = 70$
- $E \cap G$ = people who visited both England and Germany $\rightarrow n(E \cap G) = 50$
- $I \cap G$ = people who visited both Italy and Germany $\rightarrow n(I \cap G) = 30$
- $E \cap I \cap G$ = people who visited all these three countries $\rightarrow n(E \cap I \cap G) = 20$

c) How many went to none of these three countries? $\rightarrow n(E^c \cap I^c \cap G^c) = ?$

$$n(E^c \cap I^c \cap G^c) = n((E \cup I \cup G)^c) = 200 - n(E \cup I \cup G) = 200 - n[E \cup (I - E) \cup (G - (E \cup I))]$$

$$= 200 - [n(E) + n(I - E) + n(G - (E \cup I))] = 200 - [n(E) + (n(I) - n(I \cap E)) + n(G) - n(G \cap (E \cup I))]$$

$$= 200 - [n(E) + (n(I) - n(I \cap E)) + n(G) - n[(G \cap E) \cup (G \cap I)]]$$

$$= 200 - [n(E) + (n(I) - n(I \cap E)) + n(G) - n(G \cap E) - n(G \cap I) + n(E \cap I \cap G)]$$

$$= 200 - [142 + 95 + 65 - 50 - 30 - 50 + 20] = 28.$$

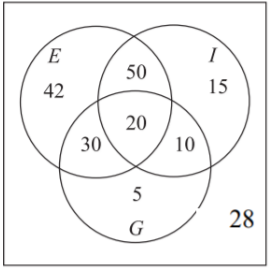
Examples- Training

Problem 3-Solution:

e) Compute the probabilities of the events described in a), b) and c)

$|\Omega| = 200, \quad n(a) = 42, \quad n(b) = 62, \quad n(c) = 28.$

$$\mathbb{P}(a) = \frac{n(a)}{|\Omega|} = \frac{42}{200} = 0.21, \quad \mathbb{P}(b) = \frac{n(b)}{|\Omega|} = \frac{62}{200} = 0.31, \quad \mathbb{P}(c) = \frac{n(c)}{|\Omega|} = \frac{28}{200} = 0.14$$



Axioms

1) $0 \leq \mathbb{P} \leq 1$
2) $\mathbb{P}(\Omega) = 1, \quad \mathbb{P}(\emptyset) = 0$
3) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
4) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
5) $A - B = A \cap B^c$
6) $A \cup B = A \cup (B - A)$ or $A \cup B = B \cup (A - B)$
7) $\mathbb{P}(A - B) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$ OR $\mathbb{P}(A - B) = \mathbb{P}(A \cup B) - \mathbb{P}(B)$
8) $\mathbb{P}(B - A) = \mathbb{P}(B) - \mathbb{P}(A \cap B)$ OR $\mathbb{P}(B - A) = \mathbb{P}(A \cup B) - \mathbb{P}(A)$
9) $\mathbb{P}(A) = \dfrac{|A|}{|\Omega|} \rightarrow$ for discrete law, $\quad \mathbb{P}(A) = \dfrac{\text{area}(A)}{\text{area}(\Omega)} \rightarrow$ for continuous law