Probability and Random Variables (ECE313/ECE317)

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Set-Theory

▷ **Fundamentals of Probability**

- Set theory, Probability spaces.
- Probability Axioms.
- Discrete uniform law.
- Continuous uniform law.

Probability: Set theory and Sample set

Set theory:

- The mathematical basis of probability is the theory of sets. For that we will review material already learned and introduce the notation and terminology as **set, element, union, intersection, and complement**

- A **set** is a collection of things:

$$A = \{0, 17, 46\}, \quad \Gamma = \{\Delta, \Box, \bigcirc, \Diamond\}$$

- The symbol $\in$ denotes set inclusion. Thus $x \in A$ means "x is an element of the set A."

$$0 \in A, \quad 17 \in A, \quad 46 \in A$$

- The symbol $\notin$ is the opposite of $\in$. Thus $c \notin A$ means "c is not an element of A or c is not in A."

$$15 \notin A$$

Probability: Set theory

- The notion of a "subset" describes a relationship between two sets: A is a subset of B if every member of A is also a member of B. Thus $A \subset B$ is a mathematical notation for the statement "the set A is a subset of the set B."

$$A \subset B \quad \text{if and only if} \quad \text{"for all } x \in A \Rightarrow x \in B\text{"}.$$

$$A = \{0, 17, 46\}, \quad B = \{0, 46\} \subset A, \quad B = \{0\} \subset A$$

- The definition of set's "equality" is:

$$A = B \;\text{ if and only if }\; B \subset A \;\textbf{ and }\; A \subset B.$$

$$\{0, 17, 46\} = \{17, 0, 46\} = \{46, 0, 17\} \quad \text{are all the same set.}$$

- The "null" set, or the "empty" set, is a set that has no elements. The notation for the empty set is $\emptyset \Rightarrow \emptyset$ is a subset of every set

$$A = \{\emptyset, 0, 17, 46\}$$

Probability: Set theory

- The "union" of sets A and B is the set of all elements that are either in A or in B, or in both, and it is denoted by $A \cup B$. Formally, the definition states

$$x \in A \cup B \quad \text{if and only if} \quad x \in A \ \text{ OR } \ x \in B.$$

$$A = \{0, 17, 46\}, \quad B = \{5, 24, 17, 3\} \quad \Rightarrow \quad A \cup B = \{0, 17, 46, 5, 24, 3\}$$

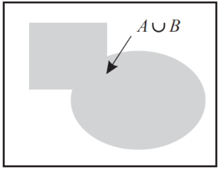


Figure: The union of sets

Probability: Set theory

- The intersection of two sets A and B is the set of all elements which are contained in both A and B, and it is denoted by $A \cap B$. Formally, the definition is

$$x \in A \cap B \quad \text{if and only if} \quad x \in A \ \ \text{AND} \ \ x \in B.$$

$$A = \{0, 17, 46\}, \quad B = \{5, 24, 17, 3\} \quad \Rightarrow \quad A \cap B = \{17\}$$

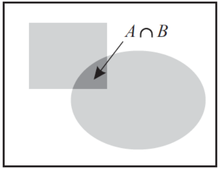


Figure: The intersection of sets

- Remark: $(A \cap B) \subset (A \cup B)$

Probability: Set theory

- The complement of a set A, denoted by A^c, is the set of all elements in the universal set (S) that are not in A. Formally,

$$x \in A^c \quad \text{if and only if} \quad x \notin A.$$

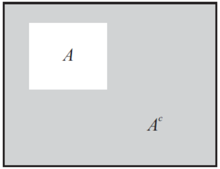


Figure: The complement of sets

$$S = \{0, 3, 12, 17, 21, 35, 40, 46, 51\}, \quad A = \{0, 17, 46\}$$
$$\Rightarrow \; A^c = \{3, 12, 21, 35, 40, 51\}$$

- We have the following properties: $A \cup A^c = S, \qquad A \cap A^c = \emptyset$

Probability: Set theory

- The difference set is a combination of intersection and complement, such that the difference between A and B is a set $A - B$ that contains all elements of A that are not elements of B. Formally,

$$x \in A - B \quad \text{if and only if} \quad x \in A \;\; \text{and} \;\; x \notin B$$

$$A = \{0, 17, 46\}, \quad B = \{5, 24, 17, 3\} \quad \Rightarrow \quad A - B = \{0, 46\}$$

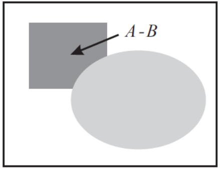
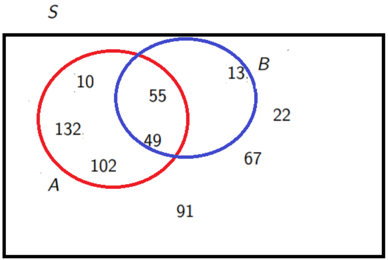


Figure: The difference of sets

Probability: Set theory

Example 1: Let the universal set $S = \{10, 13, 22, 49, 55, 132, 102, 67, 91\}$.
Let the subsets $A = \{10, 55, 132, 49, 102\}$ and $B = \{13, 49, 55\}$ of S.



- $A^c = \{13, 22, 67, 91\}$
- $B^c = \{10, 22, 132, 102, 67, 91\}$
- $A \cap B = \{55, 49\}$
- $A \cup B = \{10, 55, 132, 49, 102, 13\}$
- $A - B = \{10, 132, 102\}$.

Probability: Set theory-Properties

- We have the following properties:

1) $A - B = A \cap B^c$

- $A - B \subset A \cap B^c$?

$$x \in A - B \;\Rightarrow\; x \in A \text{ and } x \notin B$$

$$\Rightarrow\; x \in A \text{ and } x \in B^c$$

$$\Rightarrow\; x \in A \cap B^c$$

- $A \cap B^c \subset A - B$?

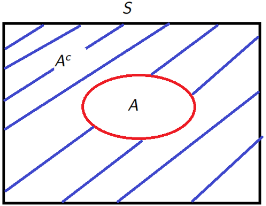
$$x \in A \cap B^c \;\Rightarrow\; x \in A \text{ and } x \in B^c$$

$$\Rightarrow\; x \in A \text{ and } x \notin B$$

$$\Rightarrow\; x \in A - B$$

Probability: Set theory

2) $A^c = S - A \quad (\text{with} \;\; A \subset S)$



- $A^c \subset S - A$?

$$x \in A^c \;\Rightarrow\; x \in S \;\text{ and }\; x \notin A$$
$$\Rightarrow\; x \in S - A$$

- $S - A \subset A^c$?

$$x \in S - A \;\Rightarrow\; x \in S \;\text{ and }\; x \notin A$$
$$\Rightarrow\; x \in A^c$$

Probability: Set theory-Properties

3) - **The DeMorgan's laws:**

$$(A \cap B)^c = A^c \cup B^c \qquad \text{and} \qquad (A \cup B)^c = A^c \cap B^c$$

- $(A \cap B)^c \subset A^c \cup B^c$?

$$\begin{aligned} x \in (A \cap B)^c \;&\Rightarrow\; x \notin (A \cap B) \\ &\Rightarrow\; x \notin A \text{ or } x \notin B \Rightarrow\; x \in A^c \text{ or } x \in B^c \\ &\Rightarrow\; x \in A^c \cup B^c \end{aligned}$$

- $A^c \cup B^c \subset (A \cap B)^c$?

$$\begin{aligned} x \in A^c \cup B^c \;&\Rightarrow\; x \in A^c \text{ or } x \in B^c \\ &\Rightarrow\; x \notin A \text{ or } x \notin B \Rightarrow\; x \notin A \cap B \\ &\Rightarrow\; x \in (A \cap B)^c \end{aligned}$$

- In the same way we can proof the second law.

Probability: Set theory-Properties

Let the sets of Example 1:
$S = \{10, 13, 22, 49, 55, 132, 102, 67, 91\}$.
$A = \{10, 55, 132, 49, 102\}$
$B = \{13, 49, 55\}$.

- $A^c = \{13, 22, 67, 91\}$
- $B^c = \{10, 22, 132, 102, 67, 91\}$
- $A \cap B = \{55, 49\}$
- $A \cup B = \{10, 55, 132, 49, 102, 13\}$
- $A - B = \{10, 132, 102\}$.

$\Rightarrow\ A^c \cup B^c = \{13, 22, 67, 91, 10, 132, 102\}$
$\Rightarrow\ (A \cap B)^c = \{10, 132, 22, 132, 102, 67, 91\}$
- We conclude that $(A \cap B)^c = A^c \cup B^c$

$\Rightarrow\ A^c \cap B^c = \{22, 67, 91\}$
$\Rightarrow\ (A \cup B)^c = \{22, 67, 91\}$
- We conclude that $(A \cup B)^c = A^c \cap B^c$

Probability: Set theory-Properties

4) $(A - B) \subset A$

$$\text{for all } \; x \in A - B \;\Rightarrow\; x \in A \;\text{ and }\; x \notin B \Rightarrow\; (A - B) \subset A$$

5) $(A^c)^c = A.$

6) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

7) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Probability: Set theory

- In working with probability we will frequently refer to two important properties of collections of sets:

mutually exclusive and **collectively exhaustive**

- A collection of sets $A_1, ..., A_n$ is mutually exclusive if and only if

$$A_i \cap A_j = \emptyset, \quad \text{for all} \quad i \neq j.$$

- If we have only two sets in the collection, we say that these sets are disjoints.

$$A \text{ and } B \text{ are disjoints if and only if } A \cap B = \emptyset.$$

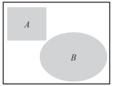


Figure: Disjoint sets

- We will use a shorthand for intersection of n sets: $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$

Probability: Set theory

- A collection of sets $A_1, ..., A_n$ is collectively exhaustive if and only if

$$A_1 \cup A_2 \cup \cdots \cup A_n = S.$$

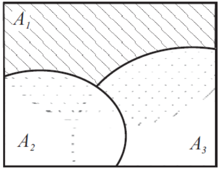


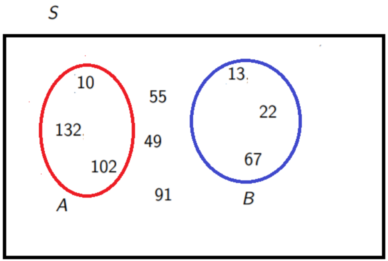
Figure: The union of sets

- We will use a shorthand for union of n sets:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n.$$

Probability: Set theory

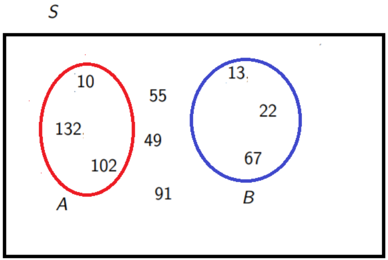
- **Example2:** Let the sets $S = \{10, 13, 22, 49, 55, 132, 102, 67, 91\}$.
$A = \{10, 132, 102\}$
$B = \{13, 22, 67\}$.



- $A^c = \{13, 22, 67, 55, 49, 91\}$
- $B^c = \{10, 132, 102, 55, 49, 91\}$
- $A \cap B = \emptyset \Rightarrow$ **A and B are disjoint**
- $A \cup B = \{10, 132, 67, 102, 13, 22\} \neq S \rightarrow$ **A and B are not collectively exhaustive**
- $A - B = A \cap B^c = A$.

Probability: Set theory

- **Example2:**



$\Rightarrow\ A^c \cup B^c = \{13, 22, 67, 55, 49, 91, 10, 132, 102\} = S$

$\Rightarrow\ (A \cap B)^c = \emptyset^c = S = \{13, 22, 67, 55, 49, 91, 10, 132, 102\}$

- Which satisfy the Morgan's law : $(A \cap B)^c = A^c \cup B^c$

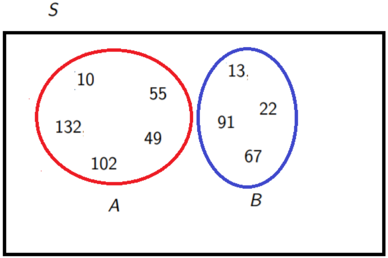
$\Rightarrow\ A^c \cap B^c = \{55, 49, 91\}$

$\Rightarrow\ (A \cup B)^c = \{55, 49, 91\}$

- Which satisfy the Morgan's law : $(A \cup B)^c = A^c \cap B^c$

Probability: Set theory

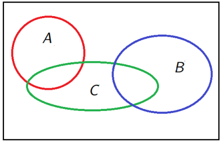
- **Example3:** Let the sets $S = \{10, 13, 22, 49, 55, 132, 102, 67, 91\}$.
$A = \{10, 132, 102, 55, 49\}$
$B = \{13, 22, 67, 91\}$.

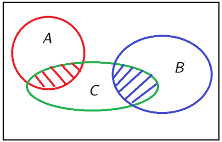


- $A^c = \{13, 22, 67, 91\} = B$
- $B^c = \{10, 132, 102, 55, 49\} = A$
- $A \cap B = \emptyset \Rightarrow$ **A and B are disjoint**
- $A \cup B = S = \{10, 13, 22, 49, 55, 132, 102, 67, 91\} \Rightarrow$ **A and B are exhaustive**.

Probability: Set theory

Remark: Consider the following sets

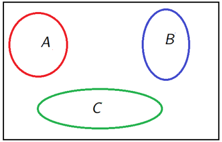




We have $A \cap C \neq \emptyset$, $B \cap C \neq \emptyset$ but $A \cap B \cap C = \emptyset$

$\Rightarrow$ A, B and c are **not mutually exclusive**.

- Consider the following sets



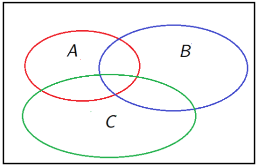
We have $A \cap C = \emptyset$, $B \cap C = \emptyset$ and $A \cap B \cap C = \emptyset$

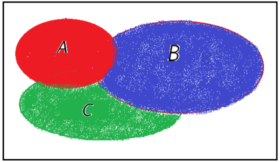
$\Rightarrow$ A, B and C are **mutually exclusive**.

Probability: Set theory

Remark1:

We can write the union $A \cup B \cup C$ as an union of disjoint sets as follow:





$$A \cup B \cup C = A \cup \ (B \cap A^c) \cup (C \cap (A^c \cap B^c)) \leftarrow$$ **union of disjoint sets**

red part   blue part   green part

- Note that the green part is computed as:

$$C - (A \cup B) = C \cap (A \cup B)^c = C \cap (A^c \cap B^c) = C \cap A^c \cap B^c$$

Probability: Set theory

Remark2:

We can write the union $A \cup B \cup C$ as an union of disjoint sets as all the following:

- $A \cup B \cup C = A \cup (C - A) \cup (B - (A \cup C)) = A \cup (C \cap A^c) \cup (B \cap (A \cup C)^c)$

$$= A \cup (C \cap A^c) \cup (B \cap (A^c \cap C^c)) = A \cup (C \cap A^c) \cup (B \cap A^c \cap C^c)$$

- $A \cup B \cup C = B \cup (A - B) \cup (C - (A \cup B)) = B \cup (A \cap B^c) \cup (C \cap (A \cup B)^c)$

$$= B \cup (A \cap B^c) \cup (C \cap (A^c \cap B^c)) = B \cup (A \cap B^c) \cup (C \cap A^c \cap B^c)$$

- $A \cup B \cup C = C \cup (B - C) \cup (A - (B \cup C)) = C \cup (B \cap C^c) \cup (A \cap (B \cup C)^c)$

$$= C \cup (B \cap C^c) \cup (A \cap (B^c \cap C^c)) = C \cup (B \cap C^c) \cup (A \cap B^c \cap C^c)$$

- We can make more combinations.

Probability: Set theory

Problem 1: Let the Figure 1:
1) Express $A \cup B$ and $A \cap B$ in different ways.
2) Express $A \cup B$ as an union of disjoints sets.

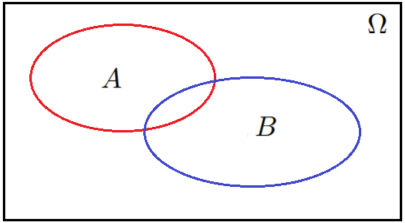


Figure 1

Probability: Set theory

Problem 1-Solution:

- 1) Let the Figure 1:

$$A \cup B = \Omega \cap (A \cup B) = (\Omega \cap A) \cup (\Omega \cap B)$$
$$A \cup B = \Omega - (A \cup B)^c = \Omega - (A^c \cap B^c)$$
$$A \cup B = A \cup (\Omega - B^c)$$
$$A \cup B = (\Omega - A^c) \cup B$$
$$A \cup B = (\Omega - A^c) \cup (\Omega - B^c)$$
$$A \cup B = (A \cap B^c) \cup (B \cap A^c) \cup (A \cap B)$$

$$A \cap B = \Omega \cap (A \cap B) = \Omega \cap A \cap B$$
$$A \cap B = \Omega - (A \cap B)^c = \Omega - (A^c \cup B^c)$$
$$A \cap B = A \cap (\Omega - B^c)$$
$$A \cap B = (\Omega - A^c) \cap B$$
$$A \cap B = (\Omega - A^c) \cap (\Omega - B^c)$$

- 2) Express $A \cup B$ as an union of disjoints sets.

$$A \cup B = A \cup (B - A) = A \cup (B \cap A^c)$$
$$A \cup B = B \cup (A - B) = B \cup (A \cap B^c)$$

Probability: Set theory

Problem 2: Write in different ways the following sets:

1) $A \cup (B \cup C)^c$
2) $[A \cup (B^c \cup C^c)]^c$
3) $[(A \cap B)^c \cup (B \cap C)]^c$

Problem 2-Solution:

1) $A \cup (B \cup C)^c = A \cup (B^c \cap C^c) = (A \cup B^c) \cap (A \cup C^c)$
2) $[A \cup (B^c \cup C^c)]^c = A^c \cap (B^c \cup C^c)^c = A^c \cap (B^c)^c \cap (C^c)^c = A^c \cap B \cap C$
3) $[(A \cap B)^c \cup (B \cap C)]^c = (A \cap B) \cap (B \cap C)^c = (A \cap B) \cap (B^c \cup C^c)$

$$= (A \cap B \cap B^c) \cup (A \cap B \cap C^c) = (A \cap \emptyset) \cup (A \cap B \cap C^c)$$
$$= \emptyset \cup (A \cap B \cap C^c) = (A \cap B \cap C^c)$$