Homework 7 (ECE 313/ECE 317, Fall 2023):

Problem 1: (5.25 points)
Consider the following table of joint PMF of the two random variables X and Y

X/Y	1	2	3	$\mathbb{P}_X(x_i)$
1	0.2	0.1	0.25	
2	0.1	0.04	0.11	
3	0	0.1	0.1	
$\mathbb{P}_Y(y_i)$				

1) Find the marginal probabilities of each random variable X and Y.

$$\mathbb{P}(X = 1) = 0.25 + 0.2 + 0.1 = 0.55 \qquad (0.25)$$

$$\mathbb{P}(X = 2) = 0.1 + 0.04 + 0.11 = 0.25 \qquad (0.25)$$

$$\mathbb{P}(X = 3) = 0 + 0.1 + 0.1 = 0.2 \qquad (0.25)$$

$$\mathbb{P}(Y = 1) = 0.2 + 0.1 + 0 = 0.3 \qquad (0.25)$$

$$\mathbb{P}(Y = 2) = 0.1 + 0.04 + 0.1 = 0.24 \qquad (0.25)$$

$$\mathbb{P}(Y = 3) = 0.25 + 0.11 + 0.1 = 0.46 \qquad (0.25)$$

X/Y	1	2	3	$\mathbb{P}_X(x_i)$
1	0.2	0.1	0.25	0.55
2	0.1	0.04	0.11	0.25
3	0	0.1	0.1	0.2
$\mathbb{P}_Y(y_i)$	0.3	0.24	0.46	1

2) Compute the expectation and the variance of each random variable

$$\mathbb{E}(X) = \sum_{x_i} x_i \mathbb{P}_X(X = x_i) = (1\times0.55)+(2\times0.25)+(3\times0.2) = 1.65 \qquad (0.5)$$

$$V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = [\sum_{x_i} x_i^2 \mathbb{P}_X(X = x_i)] - (\mathbb{E}(X))^2$$

$$= (1^2 \times 0.55) + (2^2 \times 0.25) + (3^2 \times 0.2) - (1.65)^2 = 0.62 \qquad (0.5)$$

$$\mathbb{E}(Y) = \sum_{y_i} y_i \mathbb{P}_Y(Y = y_i) = (1\times0.3)+(2\times0.24)+(3\times0.46) = 2.16 \qquad (0.5)$$

$$V(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = [\sum_{y_i} y_i^2 \mathbb{P}_Y(Y = y_i)] - (\mathbb{E}(Y))^2$$

$$= (1^2 \times 0.3) + (2^2 \times 0.24) + (3^2 \times 0.46) - (2.16)^2 = 0.73 \qquad (0.5)$$

3) Compute the covariance and the correlation of X and Y.

$$Cov(X, Y) = \sum_{x}\sum_{y}(x - \mu_X)(y - \mu_Y)\mathbb{P}_{X,Y}(x_i, y_i) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

XY	1	2	3	4	6	9
$\mathbb{P}_{X,Y}(XY)$	0.2	0.1 + 0.1 = 0.2	0.25 + 0 = 0.25	0.04	0.11+0.1 = 0.21	0.1

$$\mathbb{E}(XY) = (1\times0.2)+(2\times0.2)+(3\times0.25)+(4\times0.04)+(6\times0.21)+(9\times0.1) = 3.67$$

$$Cov(X,Y) = \mathbb{E}(XY)-\mathbb{E}(X)\mathbb{E}(Y) = 3.67-(1.65\times2.16) = 0.106 \qquad (0.75)$$

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{V(x)\times V(Y)}} = \frac{0.106}{\sqrt{0.62}\times\sqrt{0.73}} = 0.1576 \qquad (0.5)$$

4) Are X and Y independent?
• X and Y are not independent (or dependent), since $Corr \neq 0$. (0.5)

Problem 2: (3.25 points)
Suppose you take data of stock returns from the Excelsior Corporation and the Adirondack Corporation from the years 2008 to 2012, as shown here (X represents the returns to Excelsior and Y represents the returns to Adirondack):

Year	Excelsior Corp. Annual return X	Excelsior Corp. Annual return Y
2008	1	3
2009	-2	2
2010	3	4
2011	0	6
2012	3	0

1) What are the expectation and variance of each random variable.

$$\mathbb{E}(X) = \frac{1-2+3+0+3}{5} = 1 \qquad (0.5)$$

$$V(X) = \mathbb{E}(X^2)-(\mathbb{E}(X))^2 = \frac{1^2+(-2)^2+3^2+0^2+3^2}{5}-1^2 = \frac{18}{5} = 3.6 \qquad (0.5)$$

$$\mathbb{E}(Y) = \frac{3+2+4+6+0}{5} = 3 \qquad (0.5)$$

$$V(Y) = \mathbb{E}(Y^2)-(\mathbb{E}(Y))^2 = \frac{3^2+2^2+4^2+6^2+0^2}{5} - 3^2 = 4 \qquad (0.5)$$

2) What are the covariance and correlation between the stock returns (use the formula of the population covariance and correlation)?

$$Cov(X,Y) = \frac{\sum_x \sum_y (x-\mu_X)(y-\mu_Y)}{5} = \mathbb{E}(XY)-\mathbb{E}(X)\mathbb{E}(Y)$$

Year	X Y
2008	3
2009	-4
2010	12
2011	0
2012	0

$$Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \frac{3-4+12+0+0}{5} - (1\times 3) = -0.8 \qquad (0.75)$$

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{V(x)}\times\sqrt{V(Y)}} = \frac{-0.8}{\sqrt{3.6}\times\sqrt{4}} = -0.21 \qquad (0.5)$$

Problem 3:(7.5 points)

Consider the joint PMF of the random variables X and Y

X/Y	-1	0	1	$\mathbb{P}_X(x_i)$
-1	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	
0	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{6}$	
1	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	
$\mathbb{P}_Y(y_i)$				

1) Compute the marginal probabilities (1.25)

X/Y	-1	0	1	$\mathbb{P}_X(x_i)$
-1	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$
0	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{3}$
1	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{6}$
$\mathbb{P}_Y(y_i)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	1

2) Compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$ and $\mathbb{E}[XY]$.

$$\mathbb{E}[X] = \sum_i x_i \mathbb{P}(X = x_i) = (-1\times\frac{1}{2}) + (0\times\frac{1}{3}) + (1\times\frac{1}{6}) = = \frac{-1}{3} = -0.33 \qquad (0.5)$$

$$\mathbb{E}[Y] = \sum_i y_i \mathbb{P}(Y = y_i) = (-1\times\frac{1}{6}) + (0\times\frac{1}{3}) + (1\times\frac{1}{2}) = \frac{1}{3} = 0.33 \qquad (0.5)$$

XY	-1	0	1
$\mathbb{P}(XY)$	$\frac{1}{36}+\frac{1}{4}=\frac{10}{36}$	$\frac{1}{6}+\frac{1}{9}+\frac{1}{18}+\frac{1}{18}+\frac{1}{6}=\frac{10}{18}$	$\frac{1}{12}+\frac{1}{12}=\frac{2}{12}$

$$\mathbb{E}[XY] = (-1 \times \frac{10}{36}) + (0 \times \frac{10}{18}) + (1 \times \frac{2}{12}) = \frac{-1}{9} = -0.1111 \qquad (0.5)$$

3) Are X and Y independent?
   - We can remark that for all x_i and y_i, we have

   $$\mathbb{P}(X = x_i; Y = y_i) = \mathbb{P}(X = x_i).\mathbb{P}(Y = y_i) \Rightarrow X \text{ and } Y \text{ are independent} \qquad (0.5)$$

   - Note that, even $\mathbb{E}[XY] = \mathbb{E}[X].\mathbb{E}[Y]$, but we cannot conclude the independence of X and Y.

4) Compute the correlation of X and Y

   $$Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X].\mathbb{E}[Y] = \frac{-1}{9} - (\frac{-1}{3} \times \frac{1}{3}) = 0 \Rightarrow Corr = 0 \qquad (0.5)$$

5) Compute the conditional probabilities : $\mathbb{P}(X = x_i | Y = y_i)$ for all x_i and y_i

   $$\mathbb{P}(X = x_i | Y = y_i) = \frac{\mathbb{P}(X = x_i; Y = y_i)}{\mathbb{P}(Y = y_i)} \Rightarrow \mathbb{P}(X = -1 | Y = 1) = \frac{\mathbb{P}(X = -1; Y = 1)}{\mathbb{P}(Y = 1)}$$

   - We can summarize all the conditional probabilities in the following table

X/Y	-1	0	1	$\mathbb{P}(X = x_i \mid Y = y_i)$
-1	$\frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2}$	$\frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}$	$\frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$	$\mathbb{P}(X = -1 \mid Y = y_i)$
0	$\frac{\frac{1}{18}}{\frac{1}{6}} = \frac{1}{3}$	$\frac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{3}$	$\frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$	$\mathbb{P}(X = 0 \mid Y = y_i)$
1	$\frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$	$\frac{\frac{1}{18}}{\frac{1}{3}} = \frac{1}{6}$	$\frac{\frac{1}{12}}{\frac{1}{2}} = \frac{1}{6}$	$\mathbb{P}(X = 1 \mid Y = y_i)$
$\sum \mathbb{P}(X = x_i \mid Y = y_i)$	1	1	1	

(1.25)

6) Compute $\mathbb{P}(x \leq 0 | Y \geq 0) = \frac{\mathbb{P}(X \leq 0; Y \geq 0)}{\mathbb{P}(Y \geq 0)}$

X/Y	-1	0	1	$\mathbb{P}_X(x_i)$
-1	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$
0	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{3}$
1	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{6}$
$\mathbb{P}_Y(y_i)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	1

- $\mathbb{P}(X \leq 0; Y \geq 0) = \mathbb{P}(X = \{-1, 0\}; Y = \{0, 1\})$

$= \mathbb{P}(X = -1; Y = 0) + \mathbb{P}(X = 0; Y = 0) + \mathbb{P}(X = -1; Y = 1) + \mathbb{P}(X = 0; Y = 1)$

$$= \frac{1}{6} + \frac{1}{9} + \frac{1}{4} + \frac{1}{6} = \frac{25}{36} = 0.69 \quad (0.25)$$

- $\mathbb{P}(Y \geq 0) = \mathbb{P}(Y = 0) + \mathbb{P}(Y = 1) = \frac{1}{3} + \frac{1}{2} = \frac{5}{6} = 0.83 \quad (0.25)$
- $\mathbb{P}(x \leq 0 | Y \geq 0) = \frac{\mathbb{P}(X \leq 0; Y \geq 0)}{\mathbb{P}(Y \geq 0)} = \frac{\frac{25}{36}}{\frac{5}{6} = 0.83} = \frac{5}{6} \quad (0.25)$

7) Compute $\mathbb{P}(X + Y = 0)$

- Let $X + Y = 0 \Rightarrow Y = -X \Rightarrow \begin{cases} X = -1 & \Rightarrow & Y = 1 & \rightarrow & \mathbb{P}(X = -1; Y = 1) = \frac{1}{4} \\ X = 0 & \Rightarrow & Y = 0 & \rightarrow & \mathbb{P}(X = 0; Y = 0) = \frac{1}{9} \\ X = 1 & \Rightarrow & Y = -1 & \rightarrow & \mathbb{P}(X = 1; Y = -1) = \frac{1}{36} \end{cases}$

$\mathbb{P}(X + Y = 0) = \frac{1}{4} + \frac{1}{9} + \frac{1}{36} = \frac{14}{36} = 0.388 \quad (0.5)$

8) Compute $\mathbb{E}[-X + 4Y]$ and $V(X + Y)$

- $\mathbb{E}[-X + 4Y] = -\mathbb{E}[X] + 4\mathbb{E}[Y] = -(-\frac{1}{3}) + 4(\frac{1}{3}) = \frac{5}{3} = 1.6667 \quad (0.5).$
- $V(X + Y) = V(X) + V(Y) + 2Cov(X, Y)$

$$V(X) = ((-1)^2 \times \frac{1}{2}) + (0 \times \frac{1}{3}) + (1 \times \frac{1}{6}) - (\frac{-1}{3})^2 = \frac{10}{18}$$

$$V(Y) = ((-1)^2 \times \frac{1}{6}) + (0 \times \frac{1}{3}) + (1 \times \frac{1}{2}) - (\frac{1}{3})^2 = \frac{10}{18}$$

$V(X+Y) = V(X) + V(Y) + 2Cov(X, Y) = \frac{10}{18} + \frac{10}{18} + 2(0) = \frac{20}{18} = 1.1111 \quad (0.75)$

Problem 4: (04 points)

A) Let X_i be a Bernoulli distribution with parameter $p = \frac{1}{3}$ and let $S_n = X_1 + \ldots + X_{25} \rightarrow \mathcal{B}(25, \frac{1}{3})$ (follows a binomial distribution with $n = 25$ and $p = \frac{1}{3}$).

1) Compute $\mathbb{E}[S_n]$ and $V(S_n)$

$$\mathbb{E}[S_n] = np = 25(\frac{1}{3}) = 8.33 \quad (0.5)$$

$$V(S_n) = np(1 - p) = 25(\frac{1}{3})(1 - \frac{1}{3}) = \frac{50}{9} = 5.55 \quad (0.5)$$

2) Compute the following probabilities: $\mathbb{P}(S_n \leq 5)$, $\mathbb{P}(S_n \leq 8)$ and $\mathbb{P}(S_n \leq 15)$.

$$\mathbb{P}(S_n \leq 5) = \mathbb{P}(\frac{S_n - \mathbb{E}[S_n]}{\sqrt{V(S_n)}} \leq \frac{5 - \mathbb{E}[S_n]}{\sqrt{V(S_n)}}) = \mathbb{P}(\frac{S_n - 8.33}{\sqrt{5.55}} \leq \frac{5 - 8.33}{\sqrt{5.55}}) = \mathbb{P}(Z \leq -1.41) = 0.079 \qquad (0.5)$$

$$\mathbb{P}(S_n \leq 8) = \mathbb{P}(\frac{S_n - \mathbb{E}[S_n]}{\sqrt{V(S_n)}} \leq \frac{8 - \mathbb{E}[S_n]}{\sqrt{V(S_n)}}) = \mathbb{P}(\frac{S_n - 8.33}{\sqrt{5.55}} \leq \frac{8 - 8.33}{\sqrt{5.55}}) = \mathbb{P}(Z \leq -0.14) = 0.44 \qquad (0.5)$$

$$\mathbb{P}(S_n \leq 15) = \mathbb{P}(\frac{S_n - \mathbb{E}[S_n]}{\sqrt{V(S_n)}} \leq \frac{15 - \mathbb{E}[S_n]}{\sqrt{V(S_n)}}) = \mathbb{P}(\frac{S_n - 8.33}{\sqrt{5.55}} \leq \frac{15 - 8.33}{\sqrt{5.55}}) = \mathbb{P}(Z \leq 2.83) = 0.99 \qquad (0.5)$$

B) Suppose the grades in a finite mathematics class are Normally distributed with a mean of 70 and a standard deviation of 6 ($X \to \mathcal{N}(70, 6^2)$).

a) What is the probability that a randomly selected student had a grade of at least 75?

$$\mathbb{P}(X \geq 75) = \mathbb{P}(\frac{X - \mathbb{E}[X]}{\sqrt{V(X)}} \geq \frac{75 - \mathbb{E}[X]}{\sqrt{V(X)}}) = \mathbb{P}(\frac{X - 70}{6} \geq \frac{75 - 70}{6}) = \mathbb{P}(Z \geq 0.83)$$

$$= 1 - \mathbb{P}(Z \leq 0.83) = 1 - 0.79673 = 0.203 \qquad (0.5)$$

b) What is the probability that the average grade for 10 randomly selected students was at least 75?

- Let $\bar{X} = \{\mu_1, \mu_2, \dots, \mu_k\}$ be the set of means of samples of 10 students.

$$\to \bar{X} \to \mathcal{N}(\mu, (\frac{\sigma}{\sqrt{n}})^2) = \mathcal{N}(70, (\frac{6}{\sqrt{10}})^2)$$

$$\mathbb{P}(\bar{X} \geq 75) = \mathbb{P}(\frac{\bar{X} - \mathbb{E}[\bar{X}]}{\sqrt{V(\bar{X})}} \geq \frac{75 - 70}{\frac{6}{\sqrt{10}}} = \mathbb{P}(\frac{\bar{X} - 70}{\frac{6}{\sqrt{10}}} \geq \frac{75 - 70}{\frac{6}{\sqrt{10}}}) = \mathbb{P}(Z \geq 2.64)$$

$$= 1 - \mathbb{P}(Z \leq 2.64) = 1 - 0.99585 = 0.004 \qquad (0.5)$$

c) What is the probability that the average grade for 25 randomly selected students was at least 75?

- Let $\bar{X} = \{\mu_1, \mu_2, \dots, \mu_k\}$ be the set of means of samples of 25 students.

$$\to \bar{X} \to \mathcal{N}(\mu, (\frac{\sigma}{\sqrt{n}})^2) = \mathcal{N}(70, (\frac{6}{\sqrt{25}})^2)$$

$$\mathbb{P}(\bar{X} \geq 75) = \mathbb{P}(\frac{\bar{X} - \mathbb{E}[\bar{X}]}{\sqrt{V(\bar{X})}} \geq \frac{75 - 70}{\frac{6}{\sqrt{25}}} = \mathbb{P}(\frac{\bar{X} - 70}{\frac{6}{\sqrt{25}}} \geq \frac{75 - 70}{\frac{6}{\sqrt{25}}}) = \mathbb{P}(Z \geq 4.16)$$

$$= 1 - \mathbb{P}(Z \leq 4.16) = 1 - 1 = 0 \qquad (0.5)$$