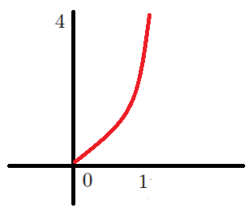

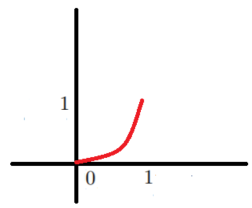

0.5 0.5

Homework 6 (ECE313/ECE317, Fall 2023):

Problem1: (7 points)
Let X be a continuous random variable that can take any value in the interval $[0, 1]$. Let its probability density function be

$$f(x) = \begin{cases} 4x^3; & x \in [0,1] \\ 0; & \text{Otherwise} \end{cases}$$

1) Plot the density function $f(x)$
2) Compute and plot the cummulative distribution $F(x)$

$$F(x) = \int_{-\infty}^{x} f(t)\ dt = \int_{0}^{x} 4t^3\ dt = [t^4]_0^x = x^4 \ldots (1.5)$$

3) What is the probability that X takes a value between $\frac{1}{2}$ and 1 (i.e; $\mathbb{P}(\frac{1}{2} < X < 1) =?$)

- **Method 1:**

$$\mathbb{P}(\frac{1}{2} < X < 1) = \mathbb{P}(X < 1) - \mathbb{P}(X \leq \frac{1}{2}) = F(1) - F(\frac{1}{2}) = 1 - \frac{1}{16} = \frac{15}{16} = 0.93 \ldots (1.5)$$

- **Method 2:**

$$\mathbb{P}(\frac{1}{2} < X < 1) = \int_{\frac{1}{2}}^{x} f(x)\ dx = \int_{\frac{1}{2}}^{1} 4x^3\ dt = [x^4]\frac{1}{2}^{1}{}^{1} = 1^4 - (\frac{1}{2})^4 = \frac{15}{16} = 0.93$$

4) What is the probability that X is less than $\frac{3}{4}$ (i.e; $\mathbb{P}(X < \frac{3}{4}) =?$).

- **Method 1:**

$$\mathbb{P}(X < \frac{3}{4}) = F(\frac{3}{4}) = (\frac{1}{16})^4 = 0.317 \ldots (1.5)$$

- **Method 2:**

$$\mathbb{P}(X < \frac{3}{4}) = \int_0^{\frac{3}{4}} f(x)\ dx = [x^4]_0^{\frac{3}{4}} = (\frac{3}{4})^4 = 0.317$$

5) What is the probability that X is greater than 0.5 (i.e; $\mathbb{P}(X > 0.7) = ?$).

$$\mathbb{P}(X > 0.7) = 1 - \mathbb{P}(X \leq 0.7) = 1 - F(0.7) = 1 - (0.7)^4 = 0.7599 \ldots (1.5)$$

Problem2: (4 points)
A man arrives at a bus stop at a random time (that is, with no regard for the scheduled service) to catch the next bus. Buses run every 30 minutes without fail, hence the next bus will come any time during the next 30 minutes with evenly distributed probability (a uniform distribution).

$$X \to \mathcal{U}(0, 30),\ \Rightarrow f(x) = \frac{1}{30} \text{ for all } x \in [0, 30] \ldots (0.5)$$

1) Find the probability that a bus will come within the next 10 minutes.

$$\mathbb{P}(X < 10) = \int_0^{10} f(x)\ dx = \int_0^{10} \frac{1}{30}\ dx = [\frac{1}{30}x]_0^{10} = \frac{10}{30} = 0.333 \ldots (1.5)$$

2) What is the average time that a bus will take to come?

$$\mathbb{E}(X) = \frac{30}{2} = 15 \text{ min.} \ldots (1.5)$$

Problem3: (4.5 points)
Heights of 25 -year-old men in a certain region have mean 67.25 inches and standard deviation 3 inches. These heights are approximately normally distributed. Thus the height X of a randomly selected 25 -year-old man is a normal random variable with mean $\mu = 67.25$ and standard deviation $\sigma = 3$.

$$X \to \mathcal{N}(\mu, \sigma^2) = \mathcal{N}(67.25, (3)^2)$$

1) Find the probability that a randomly selected 25 -year-old man is more than 69.75 inches tall.

$$\mathbb{P}(X > 67.25) = 0.5 \quad \text{since 67.25 is the mean} \ldots (1.5)$$

2) Find the probability that a randomly selected 25 -year-old man is between 60 inches and 70 inches tall.

$$\mathbb{P}(60 < X < 70) = \mathbb{P}(\frac{60-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{70-\mu}{\sigma}) = \mathbb{P}(\frac{60-67.25}{3} < Z < \frac{70-67.25}{3})$$

$$= \mathbb{P}(-2.41 < Z < 0.91) = \mathbb{P}(Z < 0.91) - \mathbb{P}(Z < -2.41) = 0.81859 - 0.00798 = 0.81065 \ldots (1.5)$$

3) Find the probability that a randomly selected 25 -year-old man is more than 80.5 inches tall.

$$\mathbb{P}(X < 80.5) = \mathbb{P}(\frac{X-\mu}{\sigma} < \frac{80.5-\mu}{\sigma}) = \mathbb{P}(Z < \frac{80.5-67.25}{3})$$

$$= \mathbb{P}(Z < 4.4167) \approx 1 \ldots (1.5)$$

Problem4: (4.5 points)
Suppose that on a certain stretch of highway, cars pass at an average rate of 6 cars per minute. Assume that the duration of time between successive cars follows the exponential distribution.

1) On average, how many **seconds** elapse between two successive cars?

The time elapse between two successive cars is $\frac{1\ min}{6} = \frac{60\ sec}{6} = 10\ sec \ldots (1.5)$

2) If the decay parameter $\lambda = \frac{1}{10}$, find the probability that after a car passes by, the next car will pass within the next 20 seconds.

$$X \to Exp(\lambda = \frac{1}{10}) \Rightarrow f(x) = \begin{cases} \lambda e^{-\lambda x}; & x > 0 \\ 0; & \text{Otherwise} \end{cases} = \begin{cases} \frac{1}{10} e^{-\frac{1}{10}x}; & x > 0 \\ 0; & \text{Otherwise} \end{cases}$$

$$\mathbb{P}(X < 20) = \int_0^{20} f(x)\, dx = \int_0^{20} \frac{1}{10} e^{-\frac{1}{10}x}\, dx = [-e^{-\frac{1}{10}x}]_0^{20} = [-e^{-\frac{20}{10}} + e^{-\frac{0}{10}}] = 0.8647 \ldots (1.5)$$

3) Consider the same $\lambda = \frac{1}{10}$, find the probability that after a car passes by, the next car will not pass for at least another 15 seconds.

$$\mathbb{P}(X > 15) = 1 - \mathbb{P}(X < 15) = 1 - \int_0^{15} f(x)\, dx = 1 - \int_0^{15} \frac{1}{10} e^{-\frac{1}{10}x}\, dx$$

$$= 1 - [-e^{-\frac{1}{10}x}]_0^{15} = 1 - [-e^{-\frac{15}{10}} + e^{-\frac{0}{10}}] = 1 - [0.286 + 1] = 0.2231 \ldots (1.5)$$