Homework 5 (ECE 313, Fall 2023):

Problem1: (2 pts)

Let u_n be an arithmetic sequence and v_n be a geometric sequence such that

$$u_n = 2 + 3n = \{u_0 = 2,\ u_1 = 5,\ u_2 = 8, \dots, u_n = 2 + 3n, \dots\}$$

$$v_n = 2 \times 3^n = \{u_0 = 2,\ u_1 = 2 \times 3^1 = 6,\ u_2 = 2 \times 3^2 = 18, \dots, u_n = 2 \times 3^n, \dots\}$$

1) $\sum_{n=0}^{10} u_n = u_0+u_1+\ldots+u_{10}+u_{11}+u_{12} = \frac{13}{2}[u_0+u_{12}] = \frac{13}{2}[2+38] = 260$ (0.5)

2) $\sum_{n=37}^{100} u_n = u_{37}+u_{38}+\ldots+u_{100} = \frac{64}{2}[u_{37}+u_{100}] = \frac{64}{2}[113+302] = 13280$ (0.5)

3) $\sum_{n=0}^{12} v_n = v_0 + v_1 + \ldots + v_{12} = 2 \times \frac{1-3^{13}}{1-3} = 1594322$ (0.5)

4) $\sum_{n=3}^{15} v_n = v_3 + v_1 + \ldots + v_{15} = v_3 \times \frac{1-3^{13}}{1-3} = 54 \times \frac{1-3^{13}}{1-3} = 43046694$ (0.5)

Problem2: (5.25 pts)

A discrete random variable X has the following probability distribution:

x_i	13	18	20	24	27
$\mathbb{P}_X(x_i)$	0.22	0.25	0.2	0.17	0.16

Compute each of the following quantities.

1) $\mathbb{P}(x < 27) = \mathbb{P}(X = 13) + \mathbb{P}(X = 18) + \mathbb{P}(X = 20) + \mathbb{P}(X = 24)$

$$= 0.22 + 0.25 + 0.2 + 0.17 = 0.84 \quad (0.5)$$

2) $\mathbb{P}(x \leq 27) = \mathbb{P}(X = 13) + \mathbb{P}(X = 18) + \mathbb{P}(X = 20) + \mathbb{P}(X = 24)$

$$+\mathbb{P}(X = 27) = 1 \quad (0.5)$$

3) $\mathbb{P}(x > 18) = 1 - \mathbb{P}(X \leq 18) = 1 - [\mathbb{P}(X = 13) + \mathbb{P}(X = 18)] = 0.53$ (0.5)

4) $\mathbb{P}(x \leq 18) = \mathbb{P}(X = 13) + \mathbb{P}(X = 18) = 0.47$ (0.5)

5) The expectation (or the mean) $\mathbb{E}(x)$

$\mathbb{E}(X) = (13 \times 0.22) + (18 \times 0.25) + (20 \times 0.2) + (24 \times 0.17) + (27 \times 0.16) = 19.76$ (0.5)

6) The expectation of X^2 (i.e; $\mathbb{E}(X^2)$)

$\mathbb{E}(X^2) = (13^2 \times 0.22) + (18^2 \times 0.25) + (20^2 \times 0.2) + (24^2 \times 0.17) + (27^2 \times 0.16) = 412.74$ (0.5)

7) The expectation of $3X+1$ (i.e; $\mathbb{E}(3X+1)$)

$$\mathbb{E}(3X+1) = 3\mathbb{E}(X)+1 = 3(19.76)+1 = 60.28 \quad (0.5)$$

8) The expectation of $2\sqrt{X}-1$ (i.e; $\mathbb{E}(2\sqrt{X}-1)$)

$$\mathbb{E}(2\sqrt{X}-1) = 2\mathbb{E}(\sqrt{X})-1 = 2[(\sqrt{13}\times 0.22)+(\sqrt{18}\times 0.25)+(\sqrt{20}\times 0.2)+(\sqrt{24}\times 0.17)+(\sqrt{27}\times 0.16)]-1$$

$$= 2(4.4125)-1 = 7.825 \quad (0.5)$$

9) The variance $V(x)$

$$V(X) = \mathbb{E}(X^2)-(\mathbb{E}(X))^2 = 412.74-(19.76)^2 = 22.28 \quad (0.5)$$

10) The standard deviation $\sigma(X)$

$$\sigma(X) = \sqrt{22.28} = 4.72 \qquad (0.25)$$

11) The variance of $3X+1$ (i.e; $V(3X+1)$).

$$V(3X+1) = 3^2V(X) = 9\times 22.28 = 200.52 \qquad (0.5)$$

Problem3: (3.5 pts)
The number X of days in the winter months that a construction crew cannot work because of the weather has the probability distribution

x_i	6	7	8	9	10	11	12	13	14
$\mathbb{P}_X(x_i)$	0.03	0.08	0.15	0.20	0.19	0.16	0.1	0.07	0.02

a) Find the probability that less than 9 days will be lost next winter.

$$\mathbb{P}(X<9) = \mathbb{P}(X=6)+\mathbb{P}(X=7)+\mathbb{P}(X=8) = 0.03+0.08+0.15 = 0.26. \quad (0.75)$$

b) Find the probability that from 8 to 12 days will be lost next winter.

$$\mathbb{P}(8\le X\le 12) = \mathbb{P}(X=8)+\mathbb{P}(X=9)+\mathbb{P}(X=10)+\mathbb{P}(X=11)+\mathbb{P}(X=12)$$

$$= 0.15+0.2+0.19+0.16+0.1 = 0.8 \qquad (0.75)$$

c) Find the probability that no more than ten days will be lost next winter.

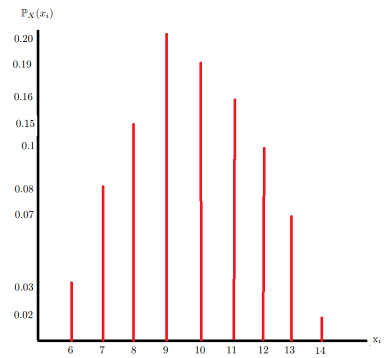
$$\mathbb{P}(X\le 10) = \mathbb{P}(X=6)+\mathbb{P}(X=7)+\mathbb{P}(X=8)+\mathbb{P}(X=9)+\mathbb{P}(X=10)$$

$$= 0.03+0.08+0.15+0.2+0.19 = 0.65. \qquad (0.75)$$

d) What is the average number of days will be lost next winter.

$$\mathbb{E}(X) = (6\times 0.03)+(7\times 0.08)+(8\times 0.15)+(9\times 0.2)+(10\times 0.19)+(11\times 0.16)+(12\times 0.1)$$

$$+(13\times 0.07)+(14\times 0.02) = 9.79. \qquad (0.75)$$



(0.5)

e) Plot this distribution.

Problem4: (2 pts)
You throw darts at a board until you hit the center area. Your probability of hitting the center area is $p = 0.53$.
X is a geometric probability $X \to \mathcal{G}(p) = \mathcal{G}(0.53) \to \mathbb{P}(X = k) = (1-p)^{k-1}p$

1) What is the probability that it takes 5 throws until you hit the center.

$$\mathbb{P}(X = 5) = (1 - 0.53)^4 \times 0.53 = 0.0259 \quad (0.75)$$

2) What is the probability that it takes less than 5 throws until you hit the center.

$$\mathbb{P}(X < 5) = \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \mathbb{P}(X = 4)$$

$$= [(1-0.53)^0 \times 0.53] + [(1-0.53)^1 \times 0.53] + [(1-0.53)^2 \times 0.53] + [(1-0.53)^3 \times 0.53] = 0.9512 \quad (0.75)$$

3) What values does X take on average?

$$\mathbb{E}(X) = \frac{1}{p} = \frac{1}{0.53} = 1.886. \quad (0.5)$$

Problem5: (2.25 pts)
A safety engineer feels that 25% of all industrial accidents in her plant are caused by failure of employees to follow instructions. She decides to look at the accident reports (selected randomly and replaced in the pile after reading) **until** she finds one that shows an accident caused by failure of employees to follow instructions.

1) What distribution does X have and what are the parameters of the distribution.
X is a geometric probability $X \to \mathcal{G}(p) = \mathcal{G}(0.25)$

$$\to \mathbb{P}(X = k) = (1-p)^{k-1}p \qquad (0.5)$$

2) On average, how many reports would the safety engineer expect to look at until she finds a report showing an accident caused by employee failure to follow instructions?

$$\mathbb{E}(X) = \frac{1}{p} = \frac{1}{0.25} = 4. \qquad (0.5)$$

3) On average, how many reports would the safety engineer expect to look at before she finds a report showing an accident caused by employee failure to follow instructions?

$$\mathbb{E}(X) = \frac{1-p}{p} = \frac{1-0.25}{0.25} = 3. \qquad (0.5)$$

4) What is the probability that the safety engineer will have to examine at least three reports until she finds a report showing an accident caused by employee failure to follow instructions?

$$\mathbb{P}(X \geq 3) = 1 - \mathbb{P}(X < 3) = \mathbb{P}(X = 1) + \mathbb{P}(X = 2)$$

$$= 1 - ([(1-0.25)^0 \times 0.25] + [(1-0.25)^1 \times 0.25]) = 0.5625. \qquad (0.75)$$

Problem6: (03 pts)
The average number of items sold by a company is 5 items per day. Let X be the random variable describing the number of items sold per day

1) What distribution does X

   X is a Poisson probability $X \to \mathcal{P}(\lambda) = \mathcal{P}(5) \to \mathbb{P}(X = k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$ (0.75)

2) What is the probability that exactly 3 items will be sold tomorrow?

$$\mathbb{P}(X = 3) = \frac{5^3 e^{-5}}{3!} = 0.14. \qquad (0.75)$$

3) What is the probability that at least 4 items will be sold tomorrow?

$$\mathbb{P}(X \geq 4) = 1 - \mathbb{P}(X < 4) = 1 - [\mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3)]$$

$$= 1 - [\frac{5^0 e^{-5}}{0!} + \frac{5^1 e^{-5}}{1!} + \frac{5^2 e^{-5}}{2!} + \frac{5^3 e^{-5}}{3!}] = 0.73. \qquad (0.75)$$

4) What is the probability that less than 5 items will be sold tomorrow?

$$\mathbb{P}(X < 5) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \mathbb{P}(X = 4)$$

$$= \frac{5^0 e^{-5}}{0!} + \frac{5^1 e^{-5}}{1!} + \frac{5^2 e^{-5}}{2!} + \frac{5^3 e^{-5}}{3!} + \frac{5^4 e^{-5}}{4!} = 0.44. \qquad (0.75)$$

Problem7: (2 pts)
Let's say that 70% of all business startups in the IT industry report that they generate a profit in their first year. If a sample of 15 new IT business startups is selected.
X is a Binomial probability $X \to \mathcal{B}(n,p) = \mathcal{B}(15, 0.7)$

$$\to \mathbb{P}(X = k) = C_k^n (1-p)^{n-k} p^k$$

1) Find the probability that exactly 10 will generate a profit in their first year.

$$\mathbb{P}(X = 10) = C_{10}^{15}(1-0.7)^5(0.7)^{10} = \frac{15!}{10!(15-10)!}(1-0.7)^5(0.7)^{10} = 0.206. \quad (01)$$

2) Find the probability that more than 7 will generate a profit in their first year

$$\mathbb{P}(X > 7) = 1 - \mathbb{P}(X \leq 7) = 1 - [\mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \mathbb{P}(X = 4)$$

$$+\mathbb{P}(X = 5) + \mathbb{P}(X = 6) + \mathbb{P}(X = 7)]$$

$$= 1 - [\frac{15!}{0!(15)!}(1-0.7)^{15}(0.7)^0 + \frac{15!}{1!(14)!}(1-0.7)^{14}(0.7)^1$$

$$+\frac{15!}{2!(13)!}(1-0.7)^{13}(0.7)^2 + \frac{15!}{3!(12)!}(1-0.7)^{12}(0.7)^3 + \frac{15!}{4!(11)!}(1-0.7)^{11}(0.7)^4$$

$$+\frac{15!}{5!(10)!}(1-0.7)^{10}(0.7)^5 + \frac{15!}{6!(9)!}(1-0.7)^9(0.7)^6 + \frac{15!}{7!(8)!}(1-0.7)^8(0.7)^7] = 1 - 0.047 = 0.953. \quad (01)$$