Homework 4-Solution (ECE 313/317, Fall 2023):

Problem 1: (4.5 points)
An urn contains 9 white and 5 black balls. We draw 5 balls **simultaneously** (i.e; without replacement and we do not take into consideration the order).

• We have 14 balls (9 W + 5 B). We draw 5 balls simultaneously $\Rightarrow$ the number of all possibilities is

$$|\Omega| = C_5^{14} = \frac{14!}{5!9!} = 2002 \text{ possible sets of 5 balls} \qquad (0.25)$$

$\rightarrow$ There is no repetition (I'm not returning back the ball)+The order doesn't matter

a) What is the probability of getting 3 whites and 2 blacks?
- Event A: {Getting 3 W and 2 B} $\rightarrow (3W, 2B)$

$$|A| = C_3^9 \times C_2^5 = \frac{9!}{3!6!} \times \frac{5!}{2!3!} = 84 \times 10 = 840 \Rightarrow \mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{840}{2002} = 0.42 \qquad (01)$$

b) What is the probability that all the 5 balls are of the same color?
- Event B: {getting all the 5 balls in the same color}
$\rightarrow$ (5 W, 0B) OR (0 W, 5B).

$$|B| = C_5^9 + C_5^5 = \frac{9!}{5!4!} + \frac{5!}{0!5!} = 127 \Rightarrow \mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{127}{2002} = 0.064.25 \qquad (01)$$

c) What is the probability of getting 1 white and 4 black balls?
- Event C: {getting 1 white and 4 black} $\rightarrow$ (1 W, 4 B)

$$|C| = C_1^9 \times C_4^5 = \frac{9!}{1!8!} \times \frac{5!}{4!1!} = 9 \times 5 = 45 \Rightarrow \mathbb{P}(C) = \frac{|C|}{|\Omega|} = \frac{45}{2002} = 0.0225 \qquad (01)$$

d) What is the probability of getting 2 white and 3 black knowing that at least one of the 5 balls is black?
• Event E: {At least one of the 5 balls is in black}

{(1 B, 4 W) or (2 B, 3 W) or (3 B, 2 W) or (4 B, 1 W) or (5 B, 0 W)}

$$|E| = (C_1^5 \times C_4^9) + (C_2^5 \times C_3^9) + (C_3^5 \times C_2^9) + (C_4^5 \times C_1^9) + (C_5^5 \times C_0^9)$$

$$= \left(\frac{5!}{1!4!} \times \frac{9!}{4!5!}\right) + \left(\frac{5!}{2!3!} \times \frac{9!}{3!6!}\right) + \left(\frac{5!}{3!2!} \times \frac{9!}{2!7!}\right) + \left(\frac{5!}{4!1!} \times \frac{9!}{1!8!}\right) + \left(\frac{5!}{5!0!} \times \frac{9!}{0!9!}\right)$$

$$(5\times126)+(10\times84)+(10\times36)+(5\times9)+(1\times1) = 1876 \Rightarrow \mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{1876}{2002} = 0.9371 \qquad (0.25)$$

- **Remark:** We can compute this probability through the complement.
• E^C = {There are no black balls} $\Leftrightarrow |E^c| = C_5^9 = 126$

$\rightarrow \mathbb{P}(E^c) = \frac{|E|}{|\Omega|} = \frac{126}{2002} = 0.0629 \rightarrow \mathbb{P}(E) = 1 - \mathbb{P}(E^c) = 1 - 0.0629 = 0.9371$

- Event D: {getting 2 white and 3 black balls} $\rightarrow$ (2 W, 3 B)

$$|D| = C_2^9 \times C_3^5 = \frac{9!}{2!7!} \times \frac{5!}{3!2!} = 36 \times 10 = 360 \Rightarrow \mathbb{P}(D) = \frac{|D|}{|\Omega|} = \frac{360}{2002} = 0.1789 \qquad (0.25)$$

- Event $D|E$: getting (2 W, 3 B) knowing that at least one of the 5 balls is black $\rightarrow$ We have $D \cap E = D$ since $D \subset E$

$$\Rightarrow \mathbb{P}(D|E) = \frac{\mathbb{P}(D \cap E)}{\mathbb{P}(E)} = \frac{\mathbb{P}(D)}{\mathbb{P}(E)} = \frac{0.1789}{0.9371} = 0.1919 = \frac{|D|}{|E|} = \frac{360}{1876} \qquad (0.75)$$

Problem 2: (3 points)
In an urn, there are 2 white balls and 3 black balls. Two balls are drawn **successively without replacement.**
• We have 5 balls (2 W + 3 B). We draw 2 balls successively without replacement $\Rightarrow$ It is an arrangement without repetition $\Rightarrow$ the # of possibilities is

$$|\Omega| = \mathbb{A}_2^5 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = 20 \text{ possible sets of 2 balls where the order matters} \qquad (0.5)$$

- Calculate the probabilities of the following two events:
1) Event A: {getting 2 balls of the same color} $\rightarrow$ (1W, 1W) or (1B,1B)

$$|A| = \mathbb{A}_2^2 + \mathbb{A}_2^3 = \frac{2!}{(2-2)!} + \frac{3!}{(3-2)!} = 2+6 = 8 \;\Rightarrow\; \mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{8}{20} = 0.4 \qquad (1.25)$$

2) Event B: {getting two balls of different colors} $\rightarrow$ (1W, 1B) or (1B, 1W) $\rightarrow$ since the order matters
• **Method 1:**

$$|B| = (\mathbb{A}_1^2 \times \mathbb{A}_1^3) + (\mathbb{A}_1^3 \times \mathbb{A}_1^2) = \left(\frac{2!}{(2-1)!} \times \frac{3!}{(3-1)!}\right) + \left(\frac{3!}{(3-1)!} \times \frac{2!}{(2-1)!}\right) = 2(2\times 3) = 12$$

$$\Rightarrow \; \mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{12}{20} = 0.6 \qquad (1.25)$$

• **Method 2:** consider it as a combination of getting (1W, 1B) but we should take into consideration all the possible permutations since the order matters

$$|B| = 2! \times (\mathbb{C}_1^2 \times \mathbb{C}_1^3) = 2! \times \frac{2!}{1!(2-1)!} \times \frac{3!}{1!(3-1)!} = 2\times 2\times 3 = 12 \;\Rightarrow\; \mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{12}{20} = 0.6$$

- We are multiplying by 2! since the order matters: If I will draw the W first and the B second is different from I'm drawing the B first and the W second.
• **Method 3:** We can go through the complement. We have $B = A^c$

$$\Rightarrow \; \mathbb{P}(B) = 1 - \mathbb{P}(A) = 1 - 0.4 = 0.6$$

Problem 3: (1.5 points)
You want to order a pizza. If you have a choice of 5 different toppings, how many different pizzas can be ordered?

$$\#\text{ of pizza } = C_0^5 + C_1^5 + C_2^5 + C_3^5 + C_4^5 + C_5^5 = 32 = 2^5 \qquad (1.5)$$

Problem 4: (4.5 points)
- How many 5-digit numbers where 0 occurs once and only once?

• We will take 5 numbers from $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, where the order matters and we have right to repeat the numbers (an arrangement with repetition).
- The first number should not be 0 We have 9 possibilities for the first number.
- We will take 0 only one time $\Rightarrow$ 0 have 4 possibilities to be placed.
- Each of the three remaining numbers have 9 possibilities.

$$\begin{array}{cccccl} \bullet & \bullet & \bullet & \bullet & \bullet & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 9 & 0 & 9 & 9 & 9 & \leftarrow \text{ possibilities when 0 is placed in the 2nd position} \\ 9 & 9 & 0 & 9 & 9 & \leftarrow \text{ possibilities when 0 is placed in the 3rd position} \\ 9 & 9 & 9 & 0 & 9 & \leftarrow \text{ possibilities when 0 is placed in the 4th position} \\ 9 & 9 & 9 & 9 & 0 & \leftarrow \text{ possibilities when 0 is placed in the 5th position} \end{array}$$

$$P = 9 \times 4 \times 9 \times 9 \times 9 = 26244. \qquad (0.75)$$

- How many ways to order the letters of MISSISSIPPI?

• It is a permutation with repetition: There are 11 letters. "S" is repeated 4 times, "I" is repeated 4 times and "P" is repeated 2 times

$$P = \frac{11!}{4!4!2!} = 34650 \qquad (0.75)$$

- How many ways are there to permute the letters of the word FORMULA?

• It is a permutation without repetition: There are 6 letters.

$$P = 7! = 5040. \qquad (0.75)$$

- How many ways are there to permute the letters of the word HAPPY?

• It is a permutation with repetition: There are 5 letters, "P" is repeated 2 times

$$P = \frac{5!}{2!} = 60. \qquad (0.75)$$

- How many ways can the letters of the word TRIANGLE be arranged if the first three letters must be RAN (in any order)?

- Let the case1: RAN $\underbrace{\{T\ I\ G\ L\ E\}}_{\text{permutes these letters}}$.

$$P = 1 \times 5! = 120. \qquad (0.75)$$

- Let the case2: $\underbrace{\{R,\ A,\ N\}}_{\text{permute these letters}} \times \underbrace{\{T\ I\ G\ L\ E\}}_{\text{permute these letters}}$.

$P = 3! \times 5! = 720$ Try to consider this answer as a correct one too

- How many different sets of four letters can be formed from the word TRIANGLE

- It is a combination of 4 elements from 8 elements.

$$P = C_4^8 = \frac{8!}{4!(8-4)!} = \frac{8!}{4!4!} = 70. \qquad (0.75)$$

Problem 5: (2 points)
In a Christmas lottery, 300 tickets are sold; 4 tickets are winners. If I buy 10 tickets, what is the probability that I will win at least one prize?

- There are C_{300}^{10} possible outcomes by buying 10 tickets from 300

$$\Rightarrow |\Omega| = C_{10}^{300} = \frac{300!}{10!(300-10)!} = \frac{300!}{10!290!} \rightarrow \text{It is a huge number} \qquad (0.5)$$

- Let the event A: I will win at least 1 prize

$$\rightarrow A = \{(1W, 9L) \text{ or } (2W, 8L) \text{ or } (3W, 7L) \text{ or } (4W, 6L)\}$$

- **Method 1:** We have 4 Winners and 296 Losing tickets

$$|A| = (C_1^4 \times C_9^{296}) + (C_2^4 \times C_8^{296}) + (C_3^4 \times C_7^{296}) + (C_4^4 \times C_6^{296})$$

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{(C_1^4 \times C_9^{296}) + (C_2^4 \times C_8^{296}) + (C_3^4 \times C_7^{296}) + (C_4^4 \times C_6^{296})}{C_{10}^{300}}$$

$$= \frac{C_1^4 \times C_9^{296}}{C_{10}^{300}} + \frac{C_2^4 \times C_8^{296}}{C_{10}^{300}} + \frac{C_3^4 \times C_7^{296}}{C_{10}^{300}} + \frac{C_4^4 \times C_6^{296}}{C_{10}^{300}} = 0.12 \qquad (1.5)$$

- Method 2: We can go through the complement.
- We have 4 W and 296 L $\Rightarrow$ The event A^c = "All the 10 tickets are losing tickets"

$$|A^c| = C_{10}^{296} \;\Rightarrow\; \mathbb{P}(A^c) = \frac{C_{10}^{296}}{C_{10}^{300}} = \frac{\frac{296!}{10!286!}}{\frac{300!}{10!290!}} = \frac{296!290!}{286!300!} = 0.8726.$$

$\rightarrow$ Less computation than the first method.

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - 0.8726 = 0.12.$$

- Method 3: We can use the binomial probability:
- Consider that we have 10 experiments with two possible results (W,L) $\rightarrow$ $\mathbb{P}(W) = \dfrac{4}{300}$ and $\mathbb{P}(L) = \dfrac{296}{300}$
Let X:{the number of the winner tickets}

$$\mathbb{P}(X = 0) = C_0^{10}[\mathbb{P}(W)]^0[\mathbb{P}(L)]^{10} = 1 \times 1 \times [\frac{296}{300}]^{10} = [\frac{296}{300}]^{10} = 0.8744$$

$$\mathbb{P}(A) = 1 - \mathbb{P}(X = 0) = 1 - 0.8744 = 0.12$$

Problem 6: (1.5 points) (Hoosier Lottery)
When you buy a Powerball ticket, you select 5 different white numbers from among the numbers 1 through 59 (order of selection does not matter), and one red number from among the numbers 1 through 35. How many different Powerball tickets can you buy?

• If you check out the Powerball you will see that you need to select 5 distinct white numbers $\rightarrow$ you can do this as follow

$$C_5^{59} = \frac{59!}{5!(59-5)!} = \frac{59!}{5!54!} = 5006386 \text{ ways.}$$

- Then you can pick the red number

$$C_1^{35} = \frac{35!}{1!(35-1)!} = \frac{35!}{1!34!} = 35 \text{ ways.}$$

$\Rightarrow$ The total number of tickets is

$$C_5^{59} \times C_1^{35} = 5006386 \times 35 = 175223510. \qquad (1.5)$$

Problem 7: (3 points)
One person is playing 10 times a game with only two possibilities success or failure. The probability that this person succeeds in the game is $p = 0.7$ and the probability of failure is $1 - p = 0.3$.
- Let the event A={The person succeeds the game 6 times}

1) What is the probability that this person will succeed 6 times in the game?
   - A typical outcome : $\{SFFSSFSFSS\}$ $\rightarrow$ This is a binomial probability.

$$\mathbb{P}(X = 6) = C_6^{10}p^6(1-p)^4 = \frac{10!}{6!4!}(0.7)^6(0.3)^4 = 210 \times 0.1176 \times 0.0081 = 0.2 \qquad (01)$$

2) If the probability of success is p = 0.5, compute in two different ways the probability that this person succeeds 6 times the game.

• **Method1:** We can use the binomial probability

$$\mathbb{P}(X=6) = C_6^{10} p^6 (1-p)^4 = \frac{10!}{6!4!}(0.5)^6(0.5)^4 = 210 \times 0.1176 \times 0.0081 = 0.2051 \quad (01)$$

• **Method2:** We can use the cardinal since the success and the failure are equally likely.

$$|\Omega| = 2^{10} = 1024, \quad |A| = C_6^{10} = \frac{10!}{6!4!} = 210$$

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{210}{1024} = 0.2051 \quad (01)$$