Homework 3 (ECE 313/ ECE 317, Spring 2023):

Problem 1: (6 points)

In a population Ω, two diseases D_1 and D_2 are present in 40% and 20% respectively.It is assumed that the number of those who suffer from both diseases is negligible. We undertake a screening system of D_1 and D_2 diseases. To do this, we apply a test that works on 90% of D_1 patients, on 70% of D_2 patients, and in 10% of individuals who have neither of these two conditions.

- Event A_1:{having disease D_1} $\rightarrow \mathbb{P}(A_1) = 0.4$
- Event A_2:{having disease D_2} $\rightarrow \mathbb{P}(A_2) = 0.2$
- Event N:{having nor of these diseases} $\rightarrow \mathbb{P}(N) = 1 - (0.4 + 0.2) = 0.4$
- Event T:{The test works}

$$\mathbb{P}(T|A_1) = 0.9, \quad \mathbb{P}(T|A_2) = 0.7, \quad \mathbb{P}(T|N) = 0.1$$

1) When we randomly choose an individual ω in Ω, what is the probability that the test will work? $\rightarrow \mathbb{P}(T) =?$

$$\begin{aligned}
\mathbb{P}(T) &= \mathbb{P}(T \cap A_1) + \mathbb{P}(T \cap A_2) + \mathbb{P}(T \cap N) \\
&= \mathbb{P}(T|A_1)\mathbb{P}(A_1) + \mathbb{P}(T|A_2)\mathbb{P}(A_2) + \mathbb{P}(T|N)\mathbb{P}(N) \\
&= (0.9).(0.4) + (0.7).(0.2) + (0.1).(0.4) \\
&= 0.36 + 0.14 + 0.04 = 0.54 \qquad (1.5)
\end{aligned}$$

- We can summarize all these values in the following table

	A_1	A_2	N	Total
T	$\mathbb{P}(A_1 \cap T) = 0.36$	$\mathbb{P}(A_2 \cap T) = 0.14$	$\mathbb{P}(N \cap T) = 0.04$	0.54
T^c	$\mathbb{P}(A_1 \cap T^c) = 0.04$	$\mathbb{P}(A_2 \cap T^c) = 0.06$	$\mathbb{P}(N \cap T^c) = 0.36$	0.16
Total	0.4	0.2	0.4	1

2) Knowing that for an individual ω, the test reacted, give the probabilities:
• So that the test reacted because of the D_1 disease.

$$\mathbb{P}(A_1|T) = \frac{\mathbb{P}(A_1 \cap T)}{\mathbb{P}(T)} = \frac{\mathbb{P}(T|A_1).\mathbb{P}(A_1)}{\mathbb{P}(T)} = \frac{0.36}{0.54} = 0.6667 \qquad (1.5)$$

• So that the test reacted because of the D_2 disease.

$$\mathbb{P}(A_2|T) = \frac{\mathbb{P}(A_2 \cap T)}{\mathbb{P}(T)} = \frac{\mathbb{P}(T|A_2).\mathbb{P}(A_2)}{\mathbb{P}(T)} = \frac{0.14}{0.54} = 0.2593 \qquad (1.5)$$

• So that the test reacted when the individual was infected with neither of the two diseases D_1 and D_2.

$$\mathbb{P}(N|T) = \frac{\mathbb{P}(N \cap T)}{\mathbb{P}(T)} = \frac{\mathbb{P}(T|N).\mathbb{P}(N)}{\mathbb{P}(T)} = \frac{0.04}{0.54} = 0.0741 \qquad (1.5)$$

Problem 2: (03 points)
One way to design a spam filter is to look at the words in an email. In particular, some words are more frequent in spam emails. Suppose that we have the following information:
• 40% of emails are spam.
• 9% of spam emails contain the word "refinance".
• 0.01% of non-spam emails contain the word "refinance".
- Let the events:

- S={The email is spam} $\rightarrow \mathbb{P}(S) = 0.4 \Rightarrow \mathbb{P}(S^c) = 0.6$
- R={The email contains the word "refinance"}

- So the following probabilities are provided to us:

$$\mathbb{P}(S) = 0.4, \quad \mathbb{P}(R|S) = 0.09, \quad \mathbb{P}(R|S^c) = 0.0001$$

- Suppose that an email is checked and found to contain the word "refinance". What is the probability that the email is spam? $\rightarrow \mathbb{P}(S|R) =?$

$$\begin{aligned} \mathbb{P}(R) &= \mathbb{P}(R \cap S) + \mathbb{P}(R \cap S^c) \\ &= \mathbb{P}(R|S).\mathbb{P}(S) + \mathbb{P}(R|S^c).\mathbb{P}(S^c) \\ &= (0.09).(0.4) + (0.0001).(0.6) = 0.036 \qquad (1.5) \end{aligned}$$

$$\begin{aligned} \mathbb{P}(S|R) &= \frac{\mathbb{P}(S \cap R)}{\mathbb{P}(R)} \\ &= \frac{\mathbb{P}(R|S).\mathbb{P}(S)}{\mathbb{P}(R)} \qquad (1.5) \\ &= \frac{(0.09).(0.4)}{0.036} = 1 \rightarrow \text{ certain event since } \mathbb{P}(R|S^c) = 0.0001 \text{ is a very low probability} \end{aligned}$$

	S	S^c	Total
R	$\mathbb{P}(S \cap R) = 0.09 \times 0.4 = 0.036$	$\mathbb{P}(S^c \cap R) = 0.0001 \times 0.6 = 6.10^{-6}$	0.036
R^c	$\mathbb{P}(S \cap R^c) = (1 - 0.09) \times 0.4 = 0.364$	$\mathbb{P}(S^c \cap R^c) = (1 - 0.0001) \times 0.6 \approx 0.6$	0.964
Total	0.4	0.6	1

Problem 3: (2.5 points)
A tennis player is entitled to two attempts to make a successful throw-in.
The player succeed his first serve 65% of the time. When it fails, it succeeds the second in 90% of cases.
- We have the following events and information:

- S_1: {The player succeed his first serve} $\rightarrow \mathbb{P}(S_1) = 0.65 \Rightarrow \mathbb{P}(S_1^c) = 0.35$
- S_2: {The player succeed his second serve}

- When it fails, the player succeeds the second in 90% of cases

$$\rightarrow \ \mathbb{P}(S_2|S_1^c) = 0.8$$

- What is the probability that the player will double fault (i.e. fail twice in a row)? $\rightarrow \mathbb{P}(S_1^c \cap S_2^c) =?$

$\mathbb{P}(S_1^c \cap S_2^c) =?$

$$\begin{aligned}\mathbb{P}(S_1^c \cap S_2^c) &= \mathbb{P}(S_2^c|S_1^c).\mathbb{P}(S_1^c)\\ &= (1 - \mathbb{P}(S_2|S_1^c)).\mathbb{P}(S_1^c)\\ &= (1 - 0.9).(0.35) = 0.035 \qquad (2.5)\end{aligned}$$

Problem 4: (4.5 points)
Suppose that I have three bags that each contain 100 marbles
$\rightarrow$ The total number of marbles is 300
$\rightarrow$ The probability of each marble is: $\mathbb{P}(B_1) = \mathbb{P}(B_2) = \mathbb{P}(B_3) = \dfrac{1}{3}$
- Let the events:

- R:{The marble is red}
- W:{The marble is white}

We have the following information:

- Bag B_1 has 80 red and 20 white marbles

$$\rightarrow \mathbb{P}(R|B_1) = 0.8, \ \ \mathbb{P}(W|B_1) = 0.2$$

- Bag B_2 has 65 red and 35 white marbles

$$\rightarrow \mathbb{P}(R|B_2) = 0.65, \ \ \mathbb{P}(W|B_2) = 0.35$$

- Bag B_3 has 45 red and 55 white marbles.

$$\rightarrow \mathbb{P}(R|B_3) = 0.45, \ \ \mathbb{P}(W|B_3) = 0.55$$

	B_1	B_2	B_3	Total
R	$\mathbb{P}(R \cap B_1) = 0.8.\frac{1}{3} = 0.2667$	$\mathbb{P}(R \cap B_2) = 0.65\frac{1}{3} = 0.2167$	$\mathbb{P}(R \cap B_3) = 0.45\frac{1}{3} = 0.15$	0.6333
W	$\mathbb{P}(W \cap B_1) = 0.2\frac{1}{3} = 0.0667$	$\mathbb{P}(W \cap B_2) = 0.35\frac{1}{3} = 0.1167$	$\mathbb{P}(W \cap B_3) = 0.55\frac{1}{3} = 0.183$	0.33
Total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

- I choose one of the bags at random and then pick a marble from the chosen bag, also at random.

1) What is the probability that the chosen marble is red?

$$\begin{aligned}\mathbb{P}(R) &= \mathbb{P}(R\cap B_1)+\mathbb{P}(R\cap B_2)+\mathbb{P}(R\cap B_3)\\ &= \mathbb{P}(R|B_1).\mathbb{P}(B_1)+\mathbb{P}(R|B_2).\mathbb{P}(B_2)+\mathbb{P}(R|B_3).\mathbb{P}(B_3)\\ &= 0.8.\frac{1}{3}+0.65.\frac{1}{3}+0.45.\frac{1}{3} = 0.63 \qquad (1.25)\end{aligned}$$

2) Suppose we observed that the chosen marble is red. What is the probability that Bag B_1 was selected? $\rightarrow \mathbb{P}(B_1|R) = ?$

$$\begin{aligned}\mathbb{P}(B_1|R) &= \frac{\mathbb{P}(B_1\cap R)}{\mathbb{P}(R)}\\ &= \frac{\mathbb{P}(R|B_1).\mathbb{P}(B_1)}{\mathbb{P}(R)}\\ &= \frac{0.8\times\frac{1}{3}}{0.63} = 0.4233 \qquad (1.25).\end{aligned}$$

3) Is the picked marble being red depend on the chosen bag? (check the independence of the two events using the conditional probability)

$$\mathbb{P}(R\cap B_1) = \mathbb{P}(R|B_1).\mathbb{P}(B_1) = \frac{0.8}{3} = 0.2667 \qquad (1.25)$$

$\mathbb{P}(R).\mathbb{P}(B_1) = \frac{0.6}{3} = 0.2111 \neq \mathbb{P}(R\cap B_1) = 0.2667 \rightarrow$ These two events are dependent. (0.75)

OR

$\mathbb{P}(R|B_1) = 0.8 \neq \mathbb{P}(R) = 0.6333 \rightarrow$ These two events are dependent (or not independent).

Problem 5: (4 points)

In my town, it's rainy one third of the days. Given that it is rainy, there will be heavy traffic with probability $\frac{1}{2}$, and given that it is not rainy, there will be heavy traffic with probability $\frac{1}{4}$. If it's rainy and there is heavy traffic, I arrive late for work with probability $\frac{1}{2}$. On the other hand, the probability of being late is reduced to $\frac{1}{8}$ if it is not rainy and there is no heavy traffic. In other situations (rainy and no traffic, not rainy and traffic) the probability of being late is 0.25. The network connection is presented in Figure 1.

- Let the following events:

- R : {it is rainy }
- T : {There is a heavy traffic}

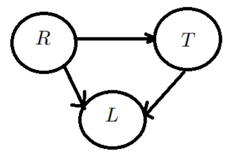

Figure 1

- L : {I'm late }

- We have the following information:

- $\mathbb{P}(R) = \frac{1}{3} \;\Rightarrow\; \mathbb{P}(R^c) = \frac{2}{3}$
- $\mathbb{P}(T|R) = \frac{1}{2} \;\Rightarrow\; \mathbb{P}(T^c|R) = 1 - \frac{1}{2} = \frac{1}{2}$
- $\mathbb{P}(T|R^c) = \frac{1}{4} \;\Rightarrow\; \mathbb{P}(T^c|R^c) = 1 - \frac{1}{4} = \frac{3}{4}$
- $\mathbb{P}(L|R \cap T) = \frac{1}{2} \;\Rightarrow\; \mathbb{P}(L^c|R \cap T) = 1 - \frac{1}{2} = \frac{1}{2}$
- $\mathbb{P}(L|R \cap T^c) = \frac{1}{4} \;\Rightarrow\; \mathbb{P}(L^c|R \cap T^c) = 1 - \frac{1}{4} = \frac{3}{4}$
- $\mathbb{P}(L|R^c \cap T) = \frac{1}{4} \;\Rightarrow\; \mathbb{P}(L^c|R^c \cap T) = 1 - \frac{1}{4} = \frac{3}{4}$
- $\mathbb{P}(L|R^c \cap T^c) = \frac{1}{8} \;\Rightarrow\; \mathbb{P}(L^c|R^c \cap T^c) = 1 - \frac{1}{8} = \frac{7}{8}$

- According to the rule of this graph, we have

$$\mathbb{P}(R \cap T \cap L) = \mathbb{P}(L|R \cap T) + \mathbb{P}(T|R) + \mathbb{P}(R)$$

- Using the above rule we can construct the following table representing all the probabilities:

	L	L^c	Total
$R \cap T$	$\mathbb{P}(L \cap R \cap T) = \frac{1}{12}$	$\mathbb{P}(L^c \cap R \cap T) = \frac{1}{12}$	$\frac{1}{6}$
$R^c \cap T$	$\mathbb{P}(L \cap R^c \cap T) = \frac{1}{24}$	$\mathbb{P}(L^c \cap R^c \cap T) = \frac{1}{8}$	$\frac{1}{6}$
$R \cap T^c$	$\mathbb{P}(L \cap R \cap T^c) = \frac{1}{24}$	$\mathbb{P}(L^c \cap R \cap T^c) = \frac{1}{8}$	$\frac{1}{6}$
$R^c \cap T^c$	$\mathbb{P}(L \cap R^c \cap T^c) = \frac{1}{16}$	$\mathbb{P}(L^c \cap R^c \cap T^c) = \frac{7}{16}$	$\frac{1}{2}$
Total	$\frac{11}{48}$	$\frac{37}{48}$	1

- We pick a random day.

1) What is the probability that it is raining and there is heavy traffic and I am late? $\rightarrow \mathbb{P}(R \cap T \cap L) =?$

$$\mathbb{P}(L \cap R \cap T) = \frac{1}{12} = 0.0833 \qquad (01)$$

2) What is the probability that it's not raining and there is heavy traffic and I am not late? $\rightarrow\ \mathbb{P}(R^c \cap T \cap L^c) =?$

$$\mathbb{P}(R^c \cap T \cap L^c) = \frac{1}{8} = 0.125 \qquad (01)$$

3) What is the probability that I am late? $\rightarrow\ \mathbb{P}(L) =?$

$$\mathbb{P}(L) = \frac{11}{48} = 0.2292 \qquad (01)$$

4) Given that I arrived late at work, what is the probability that it rained that day? $\rightarrow\ \mathbb{P}(R|L) =?$

$$\mathbb{P}(R|L) = \frac{\mathbb{P}(R \cap L)}{\mathbb{P}(L)} = \frac{\mathbb{P}(R \cap L \cap T) + \mathbb{P}(R \cap L \cap T^c)}{\mathbb{P}(L)} \ \rightarrow \text{ from the total probability}$$

$$= \frac{\frac{1}{12} + \frac{1}{24}}{\frac{11}{48}} = \frac{6}{11} = 0.5455 \qquad (01)$$