Homework 2 (ECE 313/ECE317, Fall 2023):

Problem 1: (04 points)
- Express each of the following events in term of the events A, B, and C as well as of set's operations (complement, union, and intersection):

1) At least one of the events A, B, C occurs.

$$A \cup B \cup C \qquad (0.5)$$

2) At most one of the events A, B, C occurs.

$$(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A^c \cap B^c \cap C^c) \qquad (0.5)$$

3) None of the events A, B, C occurs.

$$A^c \cap B^c \cap C^c = (A \cup B \cup C)^c \qquad (0.5)$$

4) All the three events A, B, C occur.

$$A \cap B \cap C \qquad (0.5)$$

5) Exactly one of the events A, B, C occurs.

$$(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \qquad (0.5)$$

6) Events A and B occur, but not C.

$$A \cap B \cap C^c \qquad (0.5)$$

7) Event A and B doesn't occur and C occurs.

$$(A \cap B)^c \cap C = C - (A \cap B) \qquad (0.5)$$

8) Event A occurs or, if not, then B also doesn't occur.

$$A \cup (A^c \cap B^c) \qquad (0.5)$$

Problem2: (3.75 points)
A) Let A and B be two events such that: $\mathbb{P}(B) = 0.4$, $\mathbb{P}(A \cup B) = 0.9$

1) $\mathbb{P}(A - B) = \mathbb{P}(A \cup B) - \mathbb{P}(B) = 0.9 - 0.4 = 0.5 \qquad (0.5)$

2) Let $\mathbb{P}(A \cap B) = 0.1$, we have

$$\mathbb{P}(A) = \mathbb{P}(A - B) + \mathbb{P}(A \cap B) = 0.5 + 0.1 = 0.6 \qquad (0.5)$$

3) $\mathbb{P}[(A \cap B)^c] = 1 - \mathbb{P}(A \cap B) = 1 - 0.1 = 0.9. \qquad (0.5)$

4) $\mathbb{P}(A^c \cap B^c) = \mathbb{P}[(A \cup B)^c] = 1 - \mathbb{P}(A \cup B) = 1 - 0.9 = 0.1$ (0.5)

B) Let A and B be two events. Use the axioms of probabilities to prove the following:

1) $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$.
- We have

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$$

- On the other hand, we have:

$$\begin{aligned}(A \cup B) \subset \Omega \quad &\Rightarrow \mathbb{P}(A \cup B) \leq \mathbb{P}(\Omega) = 1 \\ &\Rightarrow -\mathbb{P}(A \cup B) \geq -\mathbb{P}(\Omega) = -1\end{aligned}$$

One conclude

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1 \qquad (0.5)$$

2) $\mathbb{P}(A \cap B \cap C) \geq \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - 2$.
- We have

$$\begin{aligned}\mathbb{P}(A \cap B \cap C) &= \mathbb{P}[(A \cap B) \cap C] \\ &\geq \underbrace{\mathbb{P}(A \cap B)}_{\geq \mathbb{P}(A)+\mathbb{P}(B)-1} + \mathbb{P}(C) - 1 \rightarrow \text{ (from (1))} \\ &\geq \mathbb{P}(A) + \mathbb{P}(B) - 1 + \mathbb{P}(C) - 1 \\ &= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - 2 \qquad (0.5)\end{aligned}$$

3) The probability that one and only one of the events A or B occurs is $\mathbb{P}(A) + \mathbb{P}(B) - 2.\mathbb{P}(A \cap B)$.
- One and only one of the events A or B occurs is

$$(A \cap B^c) \cup (A^c \cap B) = (A \cup B) - (A \cap B) = (A \cup B) \cap (A \cap B)^c$$

$$\begin{aligned}\mathbb{P}[(A \cup B) \cap (A \cap B)^c] &= \mathbb{P}(A \cup B) + \mathbb{P}((A \cap B)^c) - \mathbb{P}(\underbrace{(A \cup B) \cup (A \cap B)^c}_{=\Omega}) \\ &= \mathbb{P}(A \cup B) + 1 - \mathbb{P}(A \cap B) - \mathbb{P}(\Omega) \\ &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) + 1 - \mathbb{P}(A \cap B) - 1 \\ &= \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B) \qquad (0.75)\end{aligned}$$

Problem3: (3 points)
You flip a fair coin three times.

1) What is the sample set of all the possible outcomes?

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \qquad (0.5)$$

2) What is the probability of each outcome?

$$\mathbb{P}(\text{each outcome}) = \frac{1}{8} \qquad (0.5)$$

3) What is the probability that exactly one of the flips results in a Tails?

$$A = \{THH, HTH, HHT\} \rightarrow \mathbb{P}(A) = \frac{3}{8} \qquad (0.5)$$

4) What is the probability to get at least two Heads?

$$B = \{HHT, HTH, THH, HHH\} \rightarrow \mathbb{P}(B) = \frac{4}{8} = \frac{1}{2} \qquad (0.5)$$

5) What is the probability to get at most one Tails?

$$C = \{HHT, HTH, THH, HHH\} \rightarrow \mathbb{P}(C) = \frac{4}{8} = \frac{1}{2} \qquad (0.5)$$

6) What is the probability to get the first filliping Head?

$$D = \{HTT, HHT, HTH, HHH\} \rightarrow \mathbb{P}(D) = \frac{4}{8} = \frac{1}{2} \qquad (0.5)$$

Problem4: (5.25 points)
A rectangular LCD screen has dimensions of $60cm \times 45cm$. Noting all pixels by x and y coordinates such that $0 \leq x \leq 45$ and $0 \leq y \leq 60$. Knowing that a pixel is defective, determine the probability of the following events:

1) Event A: The defective pixel is in the rectangular determined by $20 \leq x \leq 30$ and $10 \leq y \leq 50$

$$Area(A) = (30 - 20) \times (50 - 10) = 10 \times 40 = 400$$

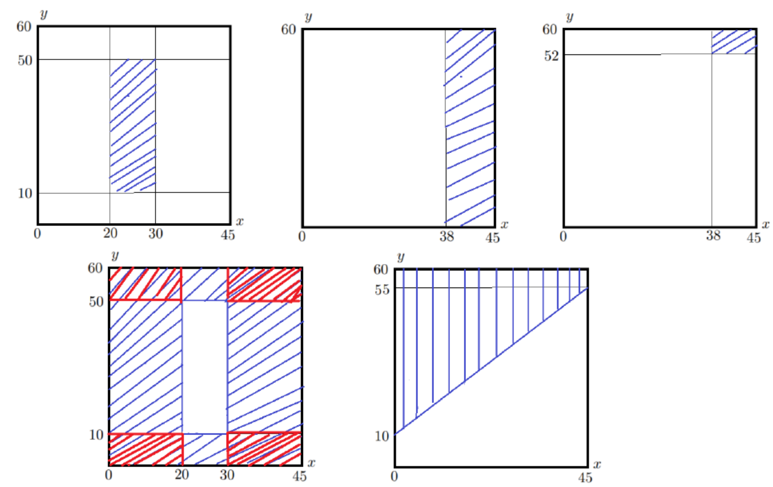
$$Area(\Omega) = 45 \times 60 = 2700 \qquad (0.25)$$

$$\mathbb{P}(A) = \frac{Area(A)}{Area(\Omega)} = \frac{400}{2700} = 0.148 \qquad (0.75)$$

2) Event B: The defective pixel is in the area determined by $x \geq 38$

$$Air(B) = (45 - 38) \times 60 = 420$$

$$\mathbb{P}(B) = \frac{Area(B)}{Area(\Omega)} = \frac{420}{2700} = 0.155 \qquad (0.75)$$



(1.25) points → (0.25) for each plot

3) Event C: The defective pixel is in the area determined by $x \geq 38$ and $y \geq 52$

$$Air(C) = (45 - 38) \times (60 - 52) = 56$$

$$\mathbb{P}(C) = \frac{Area(C)}{Area(\omega)} = \frac{56}{2700} = 0.0207 \qquad (0.75)$$

4) The defective pixel is in the area determined by $(x \leq 20 \text{ or } x \geq 30)$ and $(y \leq 10 \text{ or } y \geq 50)$

$$Area(D) = 2(15 \times 10) + 2(20 \times 10) = 300 + 400 = 700$$

$$\mathbb{P}(D) = \frac{Air(D)}{Air(\Omega)} = \frac{700}{2700} = 0.259 \qquad (0.75)$$

5) Event F: The defective pixel is on the area determined by $y - x \geq 10$.

$$Air(F) = (45 \times 5) + \frac{(55 - 10) \times 45}{2} = 1237.5$$

$$\mathbb{P}(F) = \frac{Area(F)}{Area(\Omega)} = \frac{1237.5}{2700} = 0.458 \qquad (0.75)$$

Problem5: (4 points)
In a group of 40 people, 10 people are interested in fishing, 15 in reading and 18 are not interested in fishing or reading.

- The sample set is $|S| = 40$

1) Let the event F: The person is interested in fishing $\rightarrow |F| = 10$

$$\Rightarrow \mathbb{P}(F) = \frac{|F|}{|S|} = \frac{10}{40} = 0.25 \qquad (0.5)$$

2) Let the event R: The person is interested in reading $\rightarrow |R| = 15$

$$\Rightarrow \mathbb{P}(R) = \frac{|R|}{|S|} = \frac{15}{40} = 0.375 \qquad (0.5)$$

3) Let the event C: The person is not interested in fishing or reading $\rightarrow C = (F \cup R)^c \rightarrow |C| = 18$

$$\mathbb{P}(C) = \frac{|C|}{|S|} = \frac{18}{40} = 0.45 \qquad (0.5)$$

4) The person is interested in at least one of the two activities $\rightarrow F \cup R$

$$\Rightarrow \mathbb{P}(F \cup R) = 1 - \mathbb{P}(F \cup R)^c = 1 - \mathbb{P}(C) = 1 - \frac{18}{40} = \frac{22}{40} = 0.55 \qquad (0.75)$$

5) The person is interested in both activities $\rightarrow F \cap R$.

$$\Rightarrow \mathbb{P}(F \cap R) = \mathbb{P}(F) + \mathbb{P}(R) - \mathbb{P}(F \cup R) = \frac{10}{40} + \frac{15}{40} - \frac{22}{30} = \frac{3}{40} = 0.075 \qquad (0.75)$$

6) The person is interested in only reading $\rightarrow R - (R \cap F) = R \cap (R \cap F)^c = R \cap F^c$

$$R = (R \cap F) \cup (R \cap F^c)) \Rightarrow \mathbb{P}(R) = \mathbb{P}(R \cap F) + \mathbb{P}(R \cap F^c)$$

$$\Rightarrow \mathbb{P}(R \cap F^c) = \mathbb{P}(R) - \mathbb{P}(R \cap F) = \frac{15}{40} - \frac{3}{40} = \frac{12}{40} = 0.3 \qquad (0.5)$$

7) The person is interested in only fishing $\rightarrow F - (R \cap F) = F \cap (R \cap F)^c = F \cap R^c$

$$F = (F \cap R) \cup (F \cap R^c)) \Rightarrow \mathbb{P}(F) = \mathbb{P}(F \cap R) + \mathbb{P}(F \cap R^c)$$

$$\Rightarrow \mathbb{P}(F \cap R^c) = \mathbb{P}(F) - \mathbb{P}(F \cap R) = \frac{10}{40} - \frac{3}{40} = \frac{7}{40} = 0.175 \qquad (0.5)$$