Solution-Final Exam (ECE 313, Fall 2023):

Problem1:
It is known that 25% of products on a production line are defective. Products are inspected **until** first defective is encountered.
- Let X = number of inspections to obtain first defective $\Rightarrow$ $X \to$ Geometric distribution) with parameter $p = 0.25$, such that

$$\mathbb{P}(X = k) = (1-p)^{k-1}p = (1-0.25)^{k-1}(0.25) = (0.75)^{k-1}(0.25) \ldots (0.25)$$

1) What is the probability that the first defective occurs in the sixth item inspected?

$$\mathbb{P}(X = 6) = (0.75)^5(0.25) = 0.059 \ldots (1.5)$$

2) What is the average number of inspections to obtain the first defective?

$$\mathbb{E}(x) = \frac{1}{p} = \frac{1}{0.25} = 4 \ldots (1.5)$$

3) What is the probability that the first defective occurs in the first four inspections? (i.e $\mathbb{P}(X \leq 5)$)

$$\mathbb{P}(X \leq 5) = \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \mathbb{P}(X = 4) + \mathbb{P}(X = 5)$$

$$= (0.75)^0(0.25)+(0.75)^1(0.25)+(0.75)^2(0.25)+(0.75)^3(0.25)+(0.75)^4(0.25) = 0.7627 \ldots (1.5)$$

Problem2:
A technology company uses the Poisson distribution to model the number of expected network failures per month. It has been detected that, on average, there are 3 network failures per two months (60 days)
- Let X = "The number of network failures" $\Rightarrow X \to \mathcal{P}(3)$, such that

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{3^k e^{-3}}{k!}$$

1) What is the probability that the company experiences 2 of network failures in a given two months (60 days).

$$\mathbb{P}(X = 2) = \frac{3^2 e^{-3}}{2!} = 0.224 \ldots (1.5)$$

2) What is the probability that the company experiences between 2 to 5 network failures in a given two months (60 days).

$$\mathbb{P}(2 \leq X \leq 5) = \mathbb{P}(X = 2) + \mathbb{P}(x = 3) + \mathbb{P}(X = 4) + \mathbb{P}(X = 5)$$

$$= \frac{3^2 e^{-3}}{2!} + \frac{3^3 e^{-3}}{3!} + \frac{3^4 e^{-3}}{4!} + \frac{3^5 e^{-3}}{5!} = 0.7169 \ldots (1.5)$$

3) On average, how many days elapse between two failures?

$$\frac{60}{3} = 20 \quad \text{days} \ldots (1.5)$$

4) Note that the waiting time that a network fails follows an exponential distribution with a decay parameter $\lambda = \frac{1}{20}$.

- Let X= waiting time for a network to fails $\Rightarrow X \rightarrow Exp(\lambda = \frac{1}{20})$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}; & x > 0 \\ 0; & \text{Otherwise} \end{cases} = \begin{cases} \frac{1}{20} e^{-\frac{1}{20}x}; & x > 0 \\ 0; & \text{Otherwise} \end{cases}$$

- What is the probability that a network fails within 16 days?

$$\mathbb{P}(X \leq 16) = \int_0^{16} f(x)\,dx = \int_0^{16} \frac{1}{20} e^{-\frac{1}{20}x}\,dx = [-e^{-\frac{1}{20}x}]_0^{16} = -e^{-\frac{16}{20}} + e^0 = 0.5507 \ldots (1.75)$$

Problem3: (3 points)
The distribution of retirement age for NFL players is normally distributed with a mean of 35 years old and a standard deviation of about 1.5 years.

$$f(x) = \frac{1}{2\pi} e^{\frac{-(x-\mu)^2}{\sqrt{2\sigma}}} \quad \text{with} \quad \mu = 35 \quad \text{and} \quad \sigma = 1.5$$

1) Find the probability that a randomly selected player will be retired at an age less than 33 years old?

$$\mathbb{P}(X \leq 33) = \mathbb{P}(\frac{X-35}{1.5} \leq \frac{33-35}{1.5}) = \mathbb{P}(Z \leq -1.33) = 0.0917 \ldots (1.5)$$

2) Find the probability that a randomly selected player will be retired between 33 and 35 years old.

$$\mathbb{P}(33 \leq X \leq 35) = \mathbb{P}(\frac{33-35}{1.5} \leq \frac{X-35}{1.5} \leq \frac{35-35}{1.5}) = \mathbb{P}(-1.33 \leq Z \leq 0)$$

$$= \mathbb{P}(Z \leq 0) - \mathbb{P}(Z \leq -1.33) = 0.5 - 0.0917 = 0.408 \ldots (1.5)$$

Problem4: (6 points)
Suppose you take a stock returns from the Excelsior Corporation and the Adirondack Corporation from the years 2008 to 2012, as shown here (X represents the returns to Excelsior and Y represents the returns to Adirondack):

Year	Excelsior Corp. Annual return X	Excelsior Corp. Annual return Y
2008	1	3
2009	-2	2
2010	3	4
2011	1	5
2012	2	1

1) What are the expectation and variance of each random variable.

$$\mathbb{E}(X) = \frac{1-2+3+1+2}{5} = 1 \ldots (0.75)$$

$$V(X) = \frac{1^2+(-2)^2+3^2+1^2+2^2}{5} - 1 = \frac{14}{5} = 2.8 \ldots (0.75)$$

$$\mathbb{E}(Y) = \frac{3+2+4+5+1}{5} = 3 \ldots (0.75)$$

$$V(Y) = \frac{3^2+2^2+4^2+5^2+1^2}{5} - 9 = 2 \ldots (0.75)$$

2) What are the covariance and correlation between the stock returns (use the formula of the population covariance and correlation)?

$$Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \frac{3-4+12+5+2}{5} - (3\times 1) = 0.6 \ldots (0.75)$$

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{V(X)}\sqrt{V(Y)}} = \frac{0.6}{2.3664} = 0.254 \ldots (0.25)$$

B) Let X_i be any i.i.d distributions with a mean $\mathbb{E}(X_i) = 10$ and variance $V(X_i) = (6)^2$. Let $S_n = X_1 + \ldots + X_n$

1) Which distribution follows S_n for a large n and what is their corresponding parameters?

$$S_n \to \mathcal{N}(\mathbb{E}[S_n], V(S_n)) \quad \text{a normal distribution} \quad (0.5)$$

$$\mathbb{E}(S_n) = n\mathbb{E}[X_i] = 10n \quad (0.25)$$

$$V(S_n) = nV(S_n) = n(6^2) \quad (0.25)$$

3) Compute the probability: $\mathbb{P}(S_n \geq 15)$ for $n = 6$
• For $n = 6$, $\mathbb{E}[S_n] = 60$ and $V(S_n) = 6 * (6^2) = 216$

$$\mathbb{P}(S_n \geq 15) = \mathbb{P}(\frac{S_n - \mathbb{E}[S_n]}{\sqrt{V(S_n)}} \geq \frac{15 - \mathbb{E}[S_n]}{\sqrt{V(S_n)}}) = \mathbb{P}(\frac{S_n - 60}{\sqrt{216}} \geq \frac{15-60}{\sqrt{216}}) = 1 - \mathbb{P}(Z \leq -3.06)$$

$$= 1 - 0.00013 \approx 1 \quad (0.5)$$

4) Compute the probability: $\mathbb{P}(S_n \leq 246)$ for $n = 20$
• For $n = 20$, $\mathbb{E}[S_n] = 200$ and $V(S_n) = 20 * (6^2) = 720$

$$\mathbb{P}(S_n \leq 2465) = \mathbb{P}(\frac{S_n - \mathbb{E}[S_n]}{\sqrt{V(S_n)}} \geq \frac{246 - \mathbb{E}[S_n]}{\sqrt{V(S_n)}}) = \mathbb{P}(\frac{S_n - 200}{\sqrt{720}} \geq \frac{246-200}{\sqrt{720}}) = \mathbb{P}(Z \leq 1.71) = 0.956 \ (0.5)$$