COSC 312 Homework #7 Solutions:

Use the Pumping Lemma for CFLs to prove that each of the following languages is not context-free.

1. $A = \{0^n\#0^{2n}\#0^{3n} \mid n \geq 0\}$. Hint: consider $s = 0^p\#0^{2p}\#0^{3p}$ (10 points).

   Proof. Assume A is a CFL with pumping length p and choose $s \in A$ where $s = 0^p\#0^{2p}\#0^{3p}$. We will attempt to show that $s = uvwxyz$ cannot be pumped. Now, neither v nor y can contain $\#$, otherwise $s' = uv^2xy^2z$ contains more than two $\#$'s. Therefore, if we divide s intro three segments by the $\#$'s: 0^p, 0^{2p}, and 0^{3p}, then at least one of the segments is not contained within either v or y. Hence, $s' = uv^2xy^2z$ cannot be in the language A because the $1:2:3$ length ratio of the segments of 0's cannot be maintained. Therefore, Condition 1 of the PL for CFLs is violated and our assumption that A is a CFL is incorrect. In other words, A is not a CFL. □

2. $C = \{w \mid$ the number of 1s equals the number of 2s and the number of 3s equals the number of 4s in $w\}$ and $\Sigma = \{1, 2, 3, 4\}$. Hint: consider $s = 1^p3^p2^p4^p$ (10 points).

   Proof. Assume C is a CFL with pumping length p and choose $s \in C$ where $s = 1^p3^p2^p4^p$. Because $s \in C$, it can e split $s = uvwxyz$ so that all three conditions of the PL for CFLs hold. According to Condition 3, vxy cannot contain both 1s and 2s and it cannot contain both 3s and 4s. Hence, $s' = uv^2xy^2z$ cannot contain equal numbers of 1s and 2s or equal numbers of 3s and 4s. Therefore, $s' \notin C$ and so s cannot be pumped. Condition 1 of the PL for CFLs is violated and our assumption that C is a CFL is incorrect. In other words, C is not a CFL. □

3. $B = \{w\#t \mid w$ is a substring of $t,$ where $w, t \in \{a, b\}*\}$. Hint: consider $s = a^pb^p\#a^pb^p$. (20 points)

Proof. Assume B is a CFL with pumping length p and choose $s \in B$ where $s = a^p b^p \# a^p b^p$. We will show that $s = uvxyz$ cannot be pumped. Now, neither v nor y can contain #, otherwise $s' = uv^0xy^0z$ does not contain # and we would have $s \notin B$. Four possibilities for the occurrences of v and y are enumerated below:

1. Both v and y occur on the left-hand-side of the the terminal #. In this case, $s' = uv^2xy^2z$ cannot be in B because it is longer on the left-hand-side of the #.
2. Both v and y occur on the right-hand-side of the terminal #. In this case, $s' = uv^0xy^0z$ cannot be in B because it is longer on the left-hand-side of the #.
3. One of the strings v and y is empty (we know that from Condition 3 of the PL for CFLs that both cannot be empty). Without loss of generality, we can treat them as if both occurred on the same side of the # from cases 1. and 2. above.
4. Both v and y are nonempty and straddle the terminal #. By Condition 3 of the PL for CFLs, we would have v containing only bs and y containing only as. Hence, $s' = uv^2xy^2z$ would contain more bs on the left=hand-side of the terminal # and therefore $s' \notin B$.

We have therefore shown that s cannot be pumped, i.e., Condition 1 of the PL for CFLs is violated. Since assumption that B is a CFL is incorrect, we can conclude that B is not a CFL. □

1 Grading Rubric for Proofs

1 pt: Assume C is context-free (where language is C.
1 pt: Let p be pumping length of C. 1 pt (.5 for each): Choose $s =$???? $\in C$. So $|????| \geq p$.
1 pt: By the the PL for CFLs we can be partitioned into 5 pieces $s = uvxyz$ such that for all $i \geq 0$, $uv^ixy^iz \in C$.
5 or 10 pts: t Showing and describe how the string violates the PL for CFLs (Problem 3 has more cases to consider).
1 pt: (end of proof) By this contradiction C is not CFL.