In these questions, use *alpha( )* for the inverse Ackerman function.

Question 1

Suppose I'm using Union By Height and I have a disjoint set instance with n elements. What is the running time of performing m Union operations, where $m < n$?

Question 2

Suppose I'm using Union By Rank with Path Compression, and I have a disjoint set instance with n elements. What is the running time of performing m Union and m Find operations, where $m < n$?

Question 3

Suppose that I have two STL sets, both of size n. What is the running time of creating an STL set that is their union?

Question 4

Suppose I'm using Union By Rank with Path Compression. What is the minimum number of elements that I need to have to create a set whose rank is 5?

Question 5

Suppose I'm using Union By Size. What is the minimum number of elements that I need to have to create a set whose height is 5? *Note, I'm saying "height", and not "size".*

Answers to the clicker questions

Question 1: This is $O(m)$, because union operations are $O(1)$.

Question 2: The unions are $O(1)$ each, and the finds are $O(alpha(n))$. Since that second function is bigger than the first, our running time only has to include the second. The answer is $O(m \; alpha(n))$.

Question 3: You create this set by copying one set, and then traversing the second set and inserting each element. The copy is $O(n)$, and the traversal/insertion is $O(n \; log(2n))$. That is $O(n \; log \; n)$.

Question 4: First off, in Union By Rank with Path Compression, "rank" is equivalent to "height" in Union By Height. So this question is the same as asking, "Suppose I'm using Union By Height. What is the minimum number of elements that I need to have to create a set whose height is 5."

So, let's answer a general question: Define $mh(n)$ be the minimum number of elements in a set whose height is n. We start with $n=1$. Clearly, that is 1. Now, the only time that you increase the height of a set is when you union together two sets with the same height. So, you can make a set with height n and $mh(n)$ elements by unioning two sets with $mh(n-1)$ elements. Given that:

- $mh(2) = 2*mh(1) = 2$
- $mh(3) = 2*mh(2) = 4$
- $mh(4) = 2*mh(3) = 8$
- $mh(5) = 2*mh(4) = 16$

The answer is 16.

Question 5: Now, Define $msh(n)$ be the minimum number of elements in a set, created with Union By Size, whose height is n. Again, $msh(1) = 1$. Now, suppose you have two sets, a and b, and you union them. The only time that this increases the height of either is when the height of b is greater than or equal to the height of a, and the size of a is greater than or equal to the size of b. So, to minimize $msh(n)$, you need to union a and b, where b is of height $(n-1)$, and therefore has $msh(n-1)$ elements, and a's size is greater than or equal to b's size. Therefore, a has $msh(n-1)$ elements, too.

So $msh(n) = 2*msh(n-1)$, and like question 1, the answer is 16. You should note that the logic to derive the two is different.