Question 1: 27 Points

- Part *A*: True. After either a zig-zig or zig-zag imbalance is fixed, the tree is balanced, so you can stop.
- Part *B*: False. This is backward (it's the definition of big omega).
- Part *C*: *O(n)*. Inserting into the middle of a deque is the same as inserting into the middle of a vector.
- Part *D*: *O(log n)* -- that's one reason why we like AVL trees.
- Part *E*: *O(n)* -- it's an inorder traversal.
- Part *F*: $O(n^2)$. In a sum of expressions, it's the big O of the largest expression, with any constant factors removed.
- Part *G*: *O(1)* -- that's one reason why we like lists.
- Part *H*: See part **E**. It's an inorder traversal.
- Part *I*: False. The associative array feature of a map will insert *i* if it's not there. The compiler will exit with an error if you call the method **const**.
- Part *J*: True. The very definition of big O.
- Part *K*: *O(n)*. The tree doesn't have to be balanced -- if we insert keys from "A" to "Z" in that order, the tree has a depth of 26.
- Part *L*: *O(n log(n))*. In a sum of expressions, it's the big O of the largest expression, with any constant factors removed.
- Part *M*: *O(log m)*. Inserting into a multimap is *O(log x)* where *x* is the number of elements in the multimap. Their values don't matter.
- Part *N*: True, because **b** is not modified. Now, it's a different matter as to what happens if **i** is not in the map, but that's not what the question is asking.
- Part *O*: It has to be a postorder traversal. You'll have to call a recursive clearing method on the two children, and then you delete the node. Please see **Clear()** in the lecture notes on binary search trees. It calls a postorder method called **recursive_destroy()**.
- Part *P*: This is also postorder -- you have to calculate the expressions of the children, before you can perform an arithmetic operator on them. Please see the **"Example Question"** in the lecture notes on trees.
- Part *Q*: False. Although you only need one rotation to fix a zig-zig and two rotations to fix a zig-zag, when they are fixed, you tree may still be unbalanced, so you have to keep traversing up to the root of the try to check imbalances.
- Part *R*: This is *O(n)*.

Question 2: 20 Points

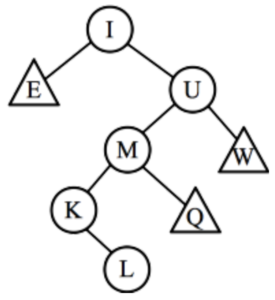
This question requires that you identify the running times of the various loops and data structures:

- **Implementation 1**: **x.size()** is *n*, and **y.size()** is *m*, and **swap()** is *O(1)*. This is *O(nm)*.
- **Implementation 2**: Pushing to the front of a deque is *O(1)*, so this is *O(n)*.
- **Implementation 3**: **x.size()** is *n*, but there are only *m* distinct elements of **x**. Therefore **s** will have a maximum of *m* elements. This is *O(n log m)*.
- **Implementation 4**: The outer loop iterates *n* times. The inner loop iterates up to *m* times (because **x's** values are between 0 and *m-1*). Finally, **F(j)** is *O(j)*. This one is $O(nm^2)$.
- **Implementation 5**: The outer loop iterates *n* times. The inner loop iterates *(log m)* times. So this is *O(n log m)*.
- **Implementation 6**: Since this is a multimap, it will have up to *n* elements. This is *O(n log n)*.
- **Implementation 7**: The loop iterates *m* times. The values of **y** are between 0 and *n-1*, and *n > m*. That means that the set can have up to *m* elements. This is *O(m log m)*.
- **Implementation 8**: The first loop is *O(n)*. The second is *O(m)*. That makes *O(n+m)*. However, we know that *m* is smaller than *n*, which means that *n+m < 2n*. Since *O(2n)* is the same as *O(n)*, the answer is *O(n)*.
- **Implementation 9**: Inserting into a the front of **v** is *O(**v.size()**)*. Therefore, this is $O(n^2)$.
- **Implementation 10**: Since there are ony *m* distinct values of **x[i]**, there are only *m* distinct values of **y[x[i]]**. This is *O(n log m)*.

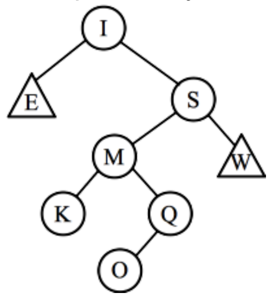
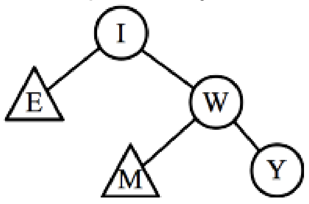
Question 3: 20 Points

- Part *A*: When we insert **Z**, node **W's** children have heights of 0 and 2. It is the lowest imbalanced node.
- Part *B*: When we insert **R**, node **Q's** children have heights of 2 and 1, so it's ok. Node **M's** children have heights of 1 and 3, so it is the lowest imbalanced node.
- Part *C*: When we delete **G**, node **E's** children have heights of 0 and 2. It is the lowest imbalanced node.
- Part *D*: When we delete **C**, nodes **A** and **E** are balanced. However, node **I** children have heights of 2 and 4, so it is imbalanced.
- Part *E*: When we delete **I**, we replace it with **G** and delete **G**. That makes **E** imbalanced (like part *C*). Alternatively, we could replace it with **K** and delete **K**. That makes **M** imbalanced. Either answer is fine.

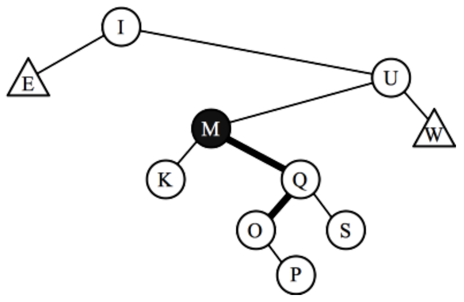
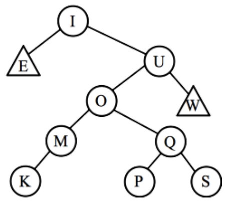
- Part *F*: **L** simply becomes the right child of **K**:



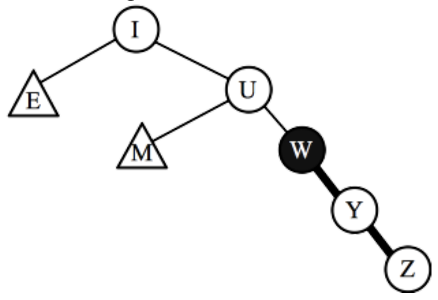
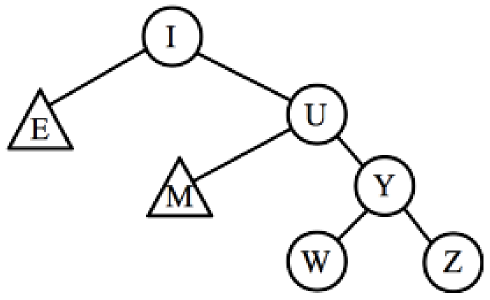
- Part *G*: When **U** is deleted, it is replaced either by the "previous" node, **S**, or the "next" node, **W**. Here are the trees in either case:

Replaced by **S**:	Replaced by **W**:
	

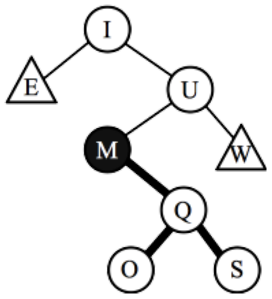
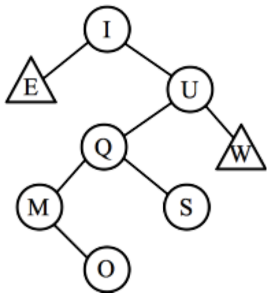
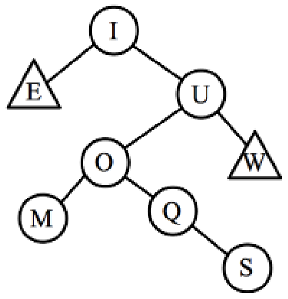
- Part H: When P is inserted, it makes node M imbalanced. It is a zig-zag imbalance, so you rotate twice about O. Below left is the imbalanced tree, and below right is after the two rotations:

After inserting **P**, node **M** is imbalanced.	Rotate twice about **O**:
	

- Part *I*: When **Z** is inserted, it makes node **W** imbalanced. It is a zig-zig imbalance, so you rotate once about **Y**. Below left is the imbalanced tree, and below right is after the rotation:

After inserting **Z**, node **W** is imbalanced.	Rotate about **Y**:
	

- Part *J*: When **K** is inserted, it makes node **M** imbalanced. It fits the case where it is both a zig-zig imbalance and a zig-zag imbalance, so you can either rotate once about **Q**, or twice about **O**. I show both below

After deleting **K**, node **M** is imbalanced.	Rotate about **Q**:	Or rotate twice about **O**:
		

Question 4: 17 Points

Straight from the lecture notes:

```
void Queue::Push(const string &s)
{
  Qnode *newnode;

  newnode = new Qnode;     // Create the new node.
  newnode->s = s;
  newnode->ptr = NULL;

  if (last == NULL) {      // If the queue is empty, set first to be this new node.
    first = newnode;
  } else {               // If the queue is non-empty, set the pointer of the last node to be this new node.
    last->ptr = newnode;
  }


  last = newnode;         // Finally, set last to point to the new node, and increment size.

  size++;
}

string Queue::Pop()
{
  Qnode *oldfirst;
  string rv;

  if (size == 0) throw((string) "Bad pop");

  /* Move "first" to point to the next node, store the return value, and
    delete the previous first node. */

  rv = first->s;
  oldfirst = first;
  first = oldfirst->ptr;
  delete oldfirst;

  /* Handle the empty queue. */

  if (first == NULL) last = NULL;

  /* Update size and return. */

  size--;
  return rv;
}
```

Question 5: 16 Points

As the writeup says, there are two parts of this -- building **v**, and then writing **rec_children()**. I'm going to start with **rec_children()**. It will have the following prototype:

```
int rec_children(const vector <int> &v, int index);
```

It will return maximum number of children of any subtree the tree that starts at **index**. As with all recursive procedures, this needs a base case -- that should be when the tree is a single node. How do we figure that out? If **v[index]** is equal to **index+1**. In that case, we return zero.

If the tree is not a single node, then it has subtrees. We need to call **rec_children()** on all of the subtrees, recording the maximum return value. At the same time, we can count the children. We return the maximum value. How do we identify the subtrees? Well:

- The first one starts at **index+1**, and ends at **v[index+1]**.
- The next one starts at **v[index+1]+1**. Call that index **i**. It ends at **v[i]**.
- The next one starts at **v[i]+1**. And so on.
- You know you're at the end when the "next" one is a right parenthesis, and not a left parenthesis. (There are other ways to figure that out. For example, if the "next" one's index is **v[index]**, you are also at the end of your children).

Here's the code for **rec_children()**:

```
int rec_children(const vector <int> &v, int index)
{
  int recursive_answer, nc, a;
  int i;

  if (v[index] == index+1) return 0;    // Base case -- if it's a single node, return 0.

  nc = 0;                                // This will hold the number of children.
  recursive_answer = 0;                  // This will hold the maximum answer for all children.

  for (i = index+1; i != v[index]; i = v[i]+1) {    // Traverse the subtrees
    nc++;                                  // Use nc to count the number of children.
    a = rec_children(v, i);                  // Compute the max children of each subtree.
    if (a > recursive_answer) recursive_answer = a;
  }
  if (nc > recursive_answer) recursive_answer = nc;
  return recursive_answer;        // Return the max of children and answers for the children.
}
```

Now, the code for **max_children()** has to create the vector **v** and then call **rec_children()**. To create the vector **v**, we use a stack (I use a deque for that) data structure. Whenever we see a left paren, we push its index onto the stack. When we see a right parent, we pop its corresponding left paren off the stack, and use that pairing to create **v**. Here's the code:

```
int max_children(const string &s)
{
  vector <int> v;
  deque <int> st;
  size_t i, j;

  v.resize(s.size(), -1);                          // Create v by using st as a stack
  for (i = 0; i < s.size(); i++) {
    if (s[i] == '(') {                           // Push the index onto the stack on '('
      st.push_front(i);
    } else {                            // Otherwise pop the index of the corresponding '('
      j = st[0];
      st.pop_front();
      v[j] = i;
    }
  }
  return rec_children(v, 0);                        // The answer is rec_children,
starting at index 0
}
```

I've put a **main()** onto this code in [q5.cpp](q5.cpp), so that you can test it. It takes the parameter string on the command line.

QR Code for q5.cpp

The majority of you did not use a stack, but instead did something like:

- Traverse **s**.
- If **s[i]** is a left paren, then search for the corresponding right paren. If you did this correctly, you had to count the number of left and right parens that you have seen, so that you can find the correct right paren.

This is an *O(n^2)* algorithm, whereas using a stack is *O(n)*. For that reason, this answer started at 5 points rather than 8.